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I. *Primitiæ Faunæ et Floræ Maderæ et Portus Sancti; sive Species quædam Novæ vel hactenus minus rite cognitæ Animalium et Plantarum in his Insulis degentium breviter descriptæ.*

CURANTE RIC. THO. LOWE, A.M.

COLL. CHR. CANT. ET NUPER AB EADEM UNIVERSITATE BACC. PERIGR.

[Read *Nov.* 15, 1830.]

SCIENTIÆ naturalis fautoribus haud quodammodo inutile futurum speravi, (obstantibus multis quominùs ***Prodromum*** meum ***Faunæ et Floræ Maderensis*** jam jamque edere possim), si specierum novarum vel hactenùs minùs cognitarum selecta quædam characteribus brevibus statim exprimere curem: adjectis annotatiunculis quibusdam, supervacaneis autem omnibus excisis, prout brevitati vel maximè consulenti oportet.

Opusculum itaque de quo nunc agitur, quasi Prodromus Prodromi, characteribus constat specificis rerum sine ordine et omninò ex arbitrio selectarum. Multa etiam nova omissa: plurima incerta alteri diei studioque accuratiori relicta sunt. Hæc præcipue de re Entomologicâ et Ichthyologicâ præmonenda velim: quippe ope in Insectis describendis, quâ prorsus confisus eram, à morte amici C. Heineken M.D., scrutatoris vel oculatissimi, orbatus sum; itaque rem omnem (in tantâ specierum novarum difficultate, entomologico vel peritissimo rem momenti non levis), radicitùs suscipere insolitus inusitatusque cogor. De Molluscis, quùm omnium terrestrium hactenùs à me repertarum catalogum completum exhibere curavi, marinas ferè omnes in præsens

omisi. De plantis Acotyledoneis veris (Cellularibus), de Crustaceis, Zoophytis, &c. idem est prædicandum. Plurima denique dubia vel nondum satis explorata in partibus omnibus consultò omisi.

Quòd si in speciebus tam zoologicis quàm botanicis novis ritè definiendis, hæc aliquantulum valeant, fructum inde percipere, haud parvum, spero; ex emendationibus, scilicet, amicis consiliisque omnium scientiæ naturalis fautorum. Rei quidem herbariæ cultoribus maximè precor ut mihi quasi *βοτανοφιλῳ* potiùs quam *βοτανοσοφῳ* indulgeant. Omnes denique rogo, quòd si in aliis corrigendis nimis aliquando videar audax, arctissimo tamen veritatis naturæque studio me semper pro viribus eniti credant: omni petulantiâ mutationisque vanæ proclivitate ab animo longè amotâ.

Amicis tantis tanta debenti, singulos enumerare, sua cuique ascribenti, locus jam non adest: nec tamen omnes silentio prætermittere possum. Cl. Rob. Brown, adjuvante J. I. Bennett arm°., summâ humanitate ac benevolentiâ, plantarum Maderensium à Masson aliisque lectarum, in Herbario Banksiano conservatarum, necnon Manuscriptorum ipsius Solandri, copiam fecit. Synonymiam certissimam Helicum à cl. Sowerby, Wood, &c. descriptarum, ex autopsiâ speciminum ipsorum, partim in Museo Britannico repostorum, benevolentiæ cl^i. Children, Gray, et G. B. Sowerby ipsius debeo. Amicus M. J. Berkeley characteres plantarum plurium novarum ab ipso in Britanniâ cultarum, propriis observationibus confirmavit, et iconibus accuratissimis illustravit. Cl^i. Hooker litteris et amicitiæ quæ quantaque non debeo! Societati denique huic illustrissimæ, si quid utilitatis, si quid commodi Scientiæ fautores ex his laboribus (vel minimum) haurire possint, gratias liceat omnium simul cum meis offerre: quippe quæ primùm inceptis nostris eximiâ liberalitate afflavit, eorum primitias illa jam priùs inde percipere debet.

PARS I^MA.

P L A N T Æ;

VASCULARES, D. C.

Class: II: MONOCOTYLEDONEÆ,

A. CRYPTOGAMÆ.

Ord. I. FILICES.

Gen. ASPIDIUM, *R. Br.*

1. Aspidium falcinellum, *Sw.*

A. fronde pinnata: foliis falcato-ensiformibus, acuminatis, argutè serrulatis, coriaceis, rigidis, petiolatis, basi sursùm obtusè auriculatis: soris biserialibus, approximatis, distinctis: indusiorum margine stellatìm dentato: stipite squamoso rhachique hirsutis.

Aspidium falcinellum, *Sw. Syn. Fil.* pp. 46, 243.

Hab. in cacuminibus montium Maderæ; locis potissimùm umbrosis, frigidis gaudens.

Cum *A. auriculato, trapezoide, Sw. &c.* species à plurimis confusa. Indusia magna, orbiculata, pulcherrima. Sori conferti, æquidistantes, haud confluentes. Nomen synonymiamque cl. Swartzii hanc speciem olìm describentis, pro novo in MSS. imposito, retinendum curavi: monentibus primùm amicis cl. Rob. Brown et J. I. Bennett armig^{ri}; quorum indicatione rem totam, ex autopsiâ speciminum in Herbario Banksiano asservatorum, luce clariorem demonstratam habeo. Swartzio *A. falcinelli* sui locus natalis planè latuit: quaproptèr species Swartziana omnibus valdè dubia; nec minùs quàm stirps maderensis cum aliis quibusdam à plurimis confusa hucusqùe visa est.

2. Aspidium frondosum, *Prodr. MS.*

A. fronde triangulari, tripinnata, subtùs hirto-paleacea, ramosa, ramis inferioribus adscendentibus: foliolis (tertii ordinis) oblongis ovatisque, acutis, inciso-dentatis, basi pinnatifidis, laciniis imis obtusis, denticulatis, imâ exteriore (superiore) majore; superioribus acutis, sub-mucronatis: soris in foliola biserialibus, demûm confluentibus: indusiis confertis, pellucidis, adpressissimis, planis, demùm marginibus reflexis: rhachibus stipiteque basi hirsutissimo pallidis, paleaceo-hirtis.

Polypodium frondosum, *Sol. MSS!*

Hab. in sylvis rupibusque siccis umbrosis Maderæ: sterile haud infrequens; cum fructu rariss.:

Frondes nitidæ, lucidæ: fertiles suprà granulatæ s. tuberculatæ. Indusia in vivo squamiformia, orbiculata, maxima, membranacea, albida, elegantia. Longit. cum stipite 2—4 pedes; Lat. 9 poll. ad 1¼ ferè pedum. Stipes 1—2 pedalis.

3. Aspidium? drepanum, *Sw.*

A? fronde lanceolatâ, acuminatâ, bipinnata: pinnis acuminatis, curvato-adscendentibus, remotiusculis: foliolis angustis, acuminatis, sub-falcatis, argutè inciso-serrulatis; inferioribus sub-oppositis; infimo superiore valdè elongato, rhachi parallelo; summis confluentibus: soris minutis, confertis, distinctis, biseriatis: rhachibus stipitibusque densè paleaceis.

Aspidium drepanum, *Sw. Syn. Fil.* pp. 54, 255.

Hab. in convallium umbrosis Maderæ; rariss.

Frondes rigidiusculæ, 2—3 pedales; pinnis 5—6 pollices, foliolis 1 pollicem ferè longis. Frondes steriles dissimillimæ, pinnis basi tantùm pinnatis, foliolisque oblongis, multò latioribus, foliosis.

Indusia non vidi. Ob fructum in omnibus exemplaribus nimis maturum, de genere dubitandum; sc. an species *Polypodiis*, *Nephrodiis*, an *Aspidiis* ritè consocianda. Indusia tamen, ut in *Aspidio caduco*, Hook. et Grev. Ic. t. 171. minutissima, vel citò caduca esse suspicor.

Gen. NEPHRODIUM, *R. Br.*

4. Nephrodium fœnisecii, *Prodr. MS.*

N. fronde triangulari vel ovata, 3—4 pinnatifida, utrinque glabra: laciniis (tertii 4-tique ordinis) oblongis, obtusis; ultimis incisis, mucronato-serratis; omnium inferioribus exterioribus internis oppositis majoribus: soris numerosis distinctis: indusiis primò semiovatis vel reniformibus, demùm orbiculatis, emarginatis: stipite breviusculo, basi sparsìm sub-paleaceo, fusco, supernè rhachique pallidis.

α. *alatum;* fronde 4–pinnatifidâ; pinnis inferioribus (1^{mi} 2^{dique} ordinis) triangularibus vel ovatis, externis interioribus oppositis valdè majoribus: pari infimo pinnarum (1^{mi} ordinis) basi deorsùm ramoso; pinnulâ (2^{di} ordinis) potissimùm 1ma (aliquando etiam 2da) inferiore s. exteriore deorsùm productâ.

Hab. in sylvis *Vaccinii padifolii, Sm.*, Maderæ; ubique vulgatissima.

β. *productum;* fronde tripinnatifidâ, paullò magìs elongatâ: pinnis omnibus oblongis; externis internis oppositis vix majoribus: laciniarum ultimarum dentibus sub-aristatis.

Hab. in umbrosis humidioribus Maderæ; rariss:

β. Status potiùs prioris (α), è loco obscuriore, defectu luminis, &c. quàm varietas videtur.

Frons in utrâque varietate nana, 1—1½ pedes (unà cum stipite) longa, ferè pedalis; 6—8 pollices lata: stipite vix dimidium totius longitudinis æquante. In utrâque odor idem gratissimus, fœnum novum redolens, constans.

Species *Aspidio dilatato* et *spinuloso Auct.* certe proxima; et cum illis forsan, in unam speciem (ut ab amiciss. cl. Hookero) consociatis, olim conjungenda. Sed distingui posse credo, figurâ frondis abbreviatâ, deltoideâ; stipite breviore, minùs (sc. basi tantùm) paleaceo; pinnulis angustioribus; odore. His adde frondem magìs decompositam: quamvis enim rarò, sc. in β, certè minùs quàm in α, decomposita, in

utroque tamen statu saltèm sub-*tri*pinnata, et longè frequentiùs, sc. in *a*, statu normali, sub-*quadri*pinnata*, Hæc omnia, cum aliis characteribus suprà indicatis, millibus exemplaribus stabilita sunt; et in plantâ *a*, adeò per totam Insulam pervulgata, constantia, nec in tantâ differentiâ loci cœlique (*β*. enim potiùs monstrosa) variantia inveni.

GEN. ASPLENIUM, *Linn. Spr. &c.*

5. Asplenium anceps, *Sol. MSS.*

A. fronde pinnata, lineari-lanceolata: pinnis distinctis, sub-petiolatis, oblongis, obtusis, apice sub-crenulatis, basi abruptis inferioribusque sursùm acutè auriculatis: soris biseriatis, obliquis, distinctis: rhachi stipiteque nitidis, fuscis, trigonis, alato-marginatis.

Asplenium anceps, *Sol. MSS!*

Hab. in Maderâ, vulgaris; *Asplenii Trichomanis* locum tenens.

GEN. GYMNOGRAMMA, *Desv. Hook.*

6. Gymnogramma Lovei, *Hook. et Grev.*

G. fronde pinnata, utrinque hirsuta: pinnis oblongis, acuminatis, pinnatifidis; summis confluentibus: laciniis ovalibus oblongisve, obtusissimis, integerrimis: stipite sparsìm squamoso rhachique hirsutis.

Gymnogr: Lovei, *Hook. et Grev. Ic. Fil. t.* 89*!*

Gymnogr: Totta, "Schlechtend: (Polypod: tottum, Willd.)" *Spr. Syst. IV.* 1. *p.* 38. *No.* 6*?*

Acrostichum pilosum, *Sol. MSS. et Herb. Banks!*

Hab. in Maderæ umbrosis humidioribus.

Pinnæ infimæ brevitèr petiolatæ; mediæ sessiles; summæ confluentes: inferiores remotiores.

---

* Ob lacinias *omnium* ordinum supernè confluentes, rectiùs 3—4 *pinnatifidæ* frondes scribuntur. Tamen ob tenuitatem suam 3—4 *pinnatæ* apparent.

Ord. II. LYCOPODIEÆ.

Gen. LYCOPODIUM, *Linn., Spr.*

7. Lycopodium suberectum, *Prodr. MS.*

L. (capsulis axillaribus): caule erecto, dichotomo; basi incurvo, decumbente: ramis fastigiatis: foliis squarrosis, 11—12 fariàm imbricatis, lineari-lanceolatis, acuminatis, rigidis, sub-pungentibus; inferioribus reflexis; superioribus erecto-patentibus.

Hab. in salebrosis fissurisque rupium sylvarum Maderæ.

Rami conferti, cespitosi, strictissimi, recti, crassitie digiti minimi, 19 ad 16 pollices alti. Species *L. Selagini* proxima; sed præter alia, longè major. Inter hanc et *L. axillare Roxb.*, et *L. Saururum Lam.* et *Bory*, quodammodò intermedia: à *L. Selagine* tamen habitu potissimùm distincta. Characteribus *L. rigido Sw.* potiùs accedit, sed habitu omninò alieno.

B. PHANEROGAMÆ.

Ord. III. GRAMINEÆ.

Gen. AIRA, *Sm.*

8. Aira argentea, *Prodr. MS.*

A. cespitosa: paniculæ coarctatæ, apice nutantis, ramis verticillatis, scabris: flosculis calycem æquantibus, basi pilosis: aristâ subdorsali, sc. ad imum ferè valvæ nascente, rectâ, flosculos duplò excedente: foliis conduplicatis, filiformibus, compressis.

Hab. in Maderæ sylvis salebrosis.

Gen. FESTUCA, *Spr.*

9. Festuca Donax, *Prodr. MS.*

F. paniculæ largæ, diffusæ, subsecundæ, nutantis ramis elongatis, flexuosis: spiculis 3–floris, lineari-lanceolatis, glomeratis; flosculis glabris,

linearibus, muticis: foliis omnibus planis, elongatis, acuminatis; marginibus scabris: culmo vaginisque glabris: ligula exserta, ovata, acuta: radice fibrosa, perenni.

Hab. in Maderæ convallibus.

Gramen giganteum, 3—4 pedale, sylvaticum.

10. Festuca albida, *Prodr. MS.*

F. densè cespitosa: paniculæ lanceolatæ, elongatæ, contractæ, erectiusculæ rhachi ramisque pubescentibus: spiculis puberulis bifloris; flosculis calyce longioribus, muticis: foliis conduplicatis, elongatis, scaberrimis, serrulatis: culmo supernè vaginisque pubescentibus; ligula abbreviata: vaginarum oris ciliatis: radice perenni.

Hab. in Maderæ convallibus.

Sylvatica, bipedalis. Culmorum bases rudes, crassissimi, perennes, glomerato-cespitosi. Folia culmos sub-æquantia, numerosa. Panicula pallida, albida. Spiculæ cum rudimento pedicellato flosculi tertii.

Habitus omninò Festucæ.

ORD. IV. CYPERACEÆ.

GEN. CAREX, *Linn., Spr.*

§§. Spicis plurimis; lateralibus androgynis, pedunculatis; terminali masculâ.

11. Carex myosuroides, *Prodr. MS.*

C. spicis ♀ (apice masculis) sub-septenis, remotissimis, solitariis, cylindricis, utrinque attenuatis, densifloris, demùm pendulis, gracilibus, elongatis, simplicibus, inferioribus pedunculatis; pedunculo vaginam duplò excedente: stigmatibus tribus: fructibus, lævibus, minimis, triquetro-oblongis, squamas lanceolatas, acuminatas æquantibus; rostro brevissimo, obtuso, integro, sæpiùs incurvo: culmo triquetro, lævi.

Hab. in Maderæ orâ septentrionali; ad margines rivulorum, scaturigines, &c.

12. Carex elata, ***Prodr. MS.***

C. spicis ♀ (apice ♂) sub-senis, remotissimis solitariis, linearibus, laxifloris, pendulis, gracilibus, elongatis, basi compositis, ramosis, omnibus inclusè pedunculatis sc. vaginâ pedunculum sub-excedente: stigmatibus tribus: fructibus costatis, obovato-triquetris, rostratis, squamas oblongo-ovatas, aristatas sub-æquantibus; rostro tenui, recto, bifido, lævi: culmo triquetro, lævi.

Hab. in Maderæ convallibus umbrosis; sylvatica.

Ord. V. ASPARAGEÆ.

Gen. ASPARAGUS, *Linn., Spr.*

13. Asparagus scoparius, ***Prodr. MS.***

A. caule frutescente, inermi, erecto, virgato ramisque patentibus, teretibus, lævibus: foliis fasciculatis, erecto-patentibus, teretibus, setaceis, lævibus, sub-mucronatis, sub-pungentibus: pedunculis densè fasciculatis, foliis sub-brevioribus.

Hab. in rupibus Maderæ.

14. Asparagus scaber, ***Prodr. MS.***

A. caule frutescente, inermi: ramis patentissimis, subdeflexis: foliis fasciculatis, rigidis, pungentibus, patentissimis, sæpe deflexis, ramulisque inæqualitèr angulatis, scabris: floribus fasciculatis; pedunculis foliorum dimidium æquantibus.

Hab. in Maderæ rupibus.

Gen. RUSCUS, *Linn., Spr.*

15. Ruscus Hypophyllum, ***Linn.***

α. latifolius; foliis ovalibus, latioribus, 7-nerviis, distichis.

R. Hypophyllum, ***Bot. Mag. t.*** 2049.

Hab. in Maderæ convallibus umbrosis.

β. *lanceolatus;* foliis lanceolatis, angustioribus, numerosis, 5–nerviis; inferioribus verticillatis; caule elatiore.

Hab. in convallibus umbrosis Maderæ.

Species forsan. Caules 2—3 pedales, superne foliosi. Folia 9—16, inferiora 5—6 pollices longa, 2—2½ lata; superiora disticha. Verticillus imus 3—6 folius. Cetera ferè ut in α.

Ord. VI. SMILACEÆ.

Gen. SMILAX, *Linn., Spr.*

16. Smilax pendulina, *Prodr. MS.*

S. caule fruticoso, scandente, sub-aculeato, tereti: aculeis caulinis raris, sparsis, abbreviatis, deflexis: foliis inermibus, coriaceis, rigidis, undulatis, 7—9 nerviis, venoso-reticulatis, latè cordatis, acuminatis; petiolis compressis, supra canaliculatis, basi 2–cirriferis: racemis flexuosis, geniculatis, longissimis, filiformibus, pendulis, terminalibus, paniculatis, ramosis; floribus ad genicula fasciculatis: baccis subglobosis, "rubris."

Smilax latifolia, *Sol. MSS. et Herb. Banks! non R. Br.*

Hab. in rupibus Maderæ.

Flores albi, racemis elegantissimis dispositi; feminei masculis paullò majores. Baccas ritè maturas non vidi; rubras incolæ ferunt.

Ord. VII. DIOSCOREÆ, *R. Br.*

Gen. TAMNUS, *Tourn., Juss.*

Tamus, *Linn.*

17. Tamnus edulis, *Prodr. MS.*

Lusitanicè, "*Norsa.*"

Anglicè, "*Porto Muniz Yam.*"

T. foliis cordato-acuminatis, 9–nerviis: stipulis sub-nullis: racemis elongatis: floribus sub-remotis; petalis ovalibus; stigmatibus simplicibus.

Dioscorea sativa, *Bowdich, Exc. in Mad. p.* 115.

Hab. in Maderâ.

Radix magna, extrinsècus pallidè brunnea, intùs alba, flavescens; sapore miti, edulis. Flores dicœci, purpurascentes, luridi. Baccæ diametro $\frac{3}{4}$ poll. ellipticæ, "rubræ." In parochiâ "Porto Muniz" dictâ, Caurum versus, solâ colitur: incolæ plantam indigenam credunt; ipse nunquam nisi planè cultam aut ex cultu ortam vidi. In Canariis, monente amico P. B. Webb, arm., procul dubio indigena; unde forsan in Maderam introducta est.

Radix in aquâ multas horas (X ad XII.) bulliente, tandem coctilis.

Ord. VIII. ORCHIDEÆ.

Gen. ORCHIS, *R. Br.*

18. Orchis foliosa, *Sol. MSS.*

O. tuberibus palmatis: labello trilobo, subplano, expanso, latiore quàm longo; lobo medio lateralibus rotundatis, crenulatis angustiore, obtuso, integro: sepalis obtusiusculis; exterioribus erectis; duobus interioribus reflexis: germine cornu descendens, tenue, æquale, obtusum superante: bracteis foliaceis, flores æquantibus: caule solido, elato.

Orchis foliosa, *Sol. MSS, Masson, et Herb. Banks!*

Hab. in umbrosis convallium sylvisque Maderæ.

Ab *Orchide longibracteatâ Bivon,* (*Bot. Reg. t.* 357): quâcum à nonnullis confusa, omninò distincta. Flores magni, purpurei, inodori. Caulis 2-pedalis.

Gen. GOODYERA, *R. Br.*

19. Goodyera macrophylla, *Prodr. MS.*——Tab. I. ff. 1—12.

G. perianthii campanulati labello glabro cochleari-calceolato; sepalis tribus exterioribus pubescentibus: columnâ anticè acuminata: massis pollinis lineari-clavatis: spicâ pubescente: floribus secundis, bracteas superantibus: foliis ovalibus, reticulato-nervosis: caule repente.

Hab. gregaria in declivibus sylvarum Maderæ humidis, umbrosis. Rariss.

Caules repentes; demùm erecti, pedales; supernè cum bracteis, germinibus, sepalisque tribus exterioribus pallidè ferrugineo-pubescentes. Folia sat magna sc. semipedalia, 3 poll. lata. Spica secundiflora, primùm pyramidata. Flores conferti, inodori, albidi, sub-cernui, $\frac{3}{8}$ poll. longi.

Class. III. DICOTYLEDONEÆ.

Ord. I. AMENTACEÆ.

Gen. SALIX, *Linn.*, *Spr.*

* *Amenta præcocia.*

20. Salix canariensis, *Sm.*

S. arborescens, ramis glaucis, pruinosis, petiolisque tomentosis: foliis lanceolatis, elongatis, utrinque attenuatis, sub-integerrimis; suprà glabriusculis, lucidis; subtùs glauco-incanis, sub-tomentosis: stipulis minutis, adpressis, ovatis, crenatis: squamis ovato-oblongis, obtusiusculis, sub-spathulatis, sericeo-villosis: germinibus magnis, pedicellatis, ovato-lanceolatis, acuminatis, styloque abbreviato glabris: stigmate utroque demùm bifido.

Hab. in rupibus madidis Maderæ: etiam Nivariæ, P. B. Webb, arm.

Arbor ferè 20 pedes alta evadit. Ramuli crassi, sæpe colorati. Gemmæ magnæ. Amenta ♂ cylindrica, abbreviata, 2-andra; ♀ elongata, graciliora. Ex characteribus videtur *S. pomeranicæ Willd.* affinis.

GEN. QUERCUS, *Linn.*

21. Quercus mitis, *Herb. Banks.*

Q. foliis ovatis, subcordatis, obtusis, integriusculis, sinuolato-denticulatis, dentibus remotis, obsoletis; subtùs, petiolis, ramulisque incano-tomentosis.

Quercus mitis, *Herb. Banks!*

Hab. in "Madera, Donne 1776:" *Herb. Banks.*

Ramuli (in specimine Banksiano) sub-umbellati; juniores tomento brevissimo, cinereo obducti. Petioli sub-semipollicares. Folia 1½—2 poll. longa; 1¼—1½ lata; alterna, ovata, obtusissima, basi sub-cordata, integriuscula s: marginibus sub-sinuolatis, nervisque lateralibus in denticulos remotos, obsoletos excurrentibus; coriacea, venosa, venis subtùs prominentibus; suprà lucida, glaberrima; subtùs cum petiolis tomento brevissimo densè velutina, in junioribus albo-incano, demùm sub-ferrugineo. Flores masculi sessiles, glomerati, in spicis abbreviatis sc: vix uncialibus, axillaribus, inferioribus congesti: Feminei pauciores, spicati, vel solitarii pedicellati, superiores sc: in axillis foliorum terminalium versus apices ramulorum supra masculos nascentes. Rhachis spicarum calycesque tomentosi vel lanuginosi.

Fructus in specimine deest.

ORD. II. URTICEÆ.

GEN. URTICA, *Linn., Spr.*

22. Urtica elevata, *Prodr. MS.*

U. caule suffruticoso, lignoso, foliisque oppositis, longè petiolatis, cordato-ovatis, grossè dentatis, lucidis, glabris: petiolis filiformibus sparsìm setosis: spicis axillaribus, pedunculatis, filiformibus, simplicibus, interruptis, paucifloris, laxis, folio multùm brevioribus.

Urtica elevata, *Herb. Banks!*

Hab. in rupibus convallium Maderæ.

Rami tenues, diffusi, debiles. Folia sat magna, ad apices ramorum sub-conferta. Habitus omninò generis. Planta inermis (haud urens sc. pungens).

GEN. PARIETARIA, *Linn.*, *Spr.*

23. Parietaria gracilis, *Prodr. MS.*

P. lucida, pubescens: caulibus ramisque gracilibus, erectis: foliis rhombeo-ovatis, obtusis, 3–nerviis, petiolatis; petiolis filiformibus, folia æquantibus: glomerulis axillaribus; floribus pedunculatis, sub-cymosis, 1—2—3 bracteatis; bracteis (sæpiùs 3) angustis, lanceolatis, calyce 4–fido brevioribus, post anthesin glanduloso-pubescentibus, inæqualibus, 1—2 dilatatis, foliaceis, calycem superantibus.

Hab. in Madera; rariss.

ORD. III. LAURINEÆ.

GEN. LAURUS, *Spr.*

24. Laurus Barbusana, *Prodr. MS.*

L. foliis perennantibus, lanceolato-oblongis, utrinque attenuatis, coriaceis, rigidis; suprà nitidissimis; infra axillis venarum nudis (e-glandulosis): pedunculis ad ramulorum apices congestis, paniculatis, sub-racemosis: pedicellis sub-elongatis, laxis: floribus hermaphroditis; calycibus sexfidis.

Hab. in Maderæ Sylvis.

Arbor magna. Folia sæpe cymbiformia. Antheræ biloculares. Drupa $\frac{3}{4}$ pollicis longa, $\frac{1}{2}$ lata; non calyculata.

ORD. IV. CHENOPODIEÆ.

GEN. ATRIPLEX, *Linn.*, *Spr.*

25. Atriplex parvifolia, *Prodr. MS.*

A. suffruticosa, procumbens, farinoso-incana: foliis confertis, alternis, ellipticis vel oblongis, repandis, sub-sinuato-erosis vel integris: valvis hastatis, integerrimis, dorso muriculato-tuberculatis.

Atriplex portulacoides (angustifolia) *Herb. Banks!*

Atriplex portulacoides var. angustifolia, *Sol. MSS!*

Hab. in Insulâ Portûs S^{a}. In Canariis, P. B. Webb, armig.

Species videtur: Cum *A. portulacoide* verâ sæpe forsan confusa. Cf. *A. portulacoiden Desf. Fl. Atl.* II. p. 392. An huc quoque spectat *A. verrucifera β. angustifolia Bieberst?*

Ord. V. NYCTAGINEÆ.

Gen. MIRABILIS. *Linn., Spr.*

26. Mirabilis divaricata, *Prodr. MSS.*

M. floribus congestis, terminalibus, sub-pedunculatis: corollâ calycem sextuplò superante; tubo longissimo, pubescente; limbo plicato (laciniis emarginatis) tubi quartam partem æquante: foliis sub-cordatis, petiolatis; suprà, petiolis, lineâque caulinâ utrinque exaratâ sub-pubescentibus: ramis dichotomis, nodosis, cauleque erectis: pericarpio rugoso, glabro, (atro).

Mirabilis hybrida, *Lepell?* sed folia in plantâ Maderensi (quamvis lucida) minimè glabra, &c.

Hab. in hortis et ruderatis Maderæ. Circa urbem Funchalensem nunc quasi indigena.

Valde ramosa, 3—5 pedalis; ramis divaricatis, demum corymbosis vel convexo-fastigiatis. In *M. Jalapá* verâ caules multo humiliores, minus ramosi; pericarpia minora, ferrugineo-pubescentia, minus rugosa, egranulata.

Ord. VI. PLANTAGINEÆ.

Gen. PLANTAGO, *Linn., Spr.*

27. Plantago leiopetala, *Prodr. MS.*

P. caulescens: caule abbreviato, basi frutescente: foliis confertis, lanceolatis, utrinque attenuatis, nervosis, glabriusculis, nitidis, integerrimis: pedunculis folia superantibus, angulatis, glabris: spicis abbreviatis,

oblongis ovatisve, obtusis, nudis: laciniis calycinis latis, scariosis, carinatis, corollisque glabris.

Hab. in cacuminibus Ins^{æ}: Portûs S^{ti}.

Plantagini lanceolatæ proxima; culturâ non mutatur.

ORD. VII. PLUMBAGINEÆ.

GEN. STATICE, *Spr.*

28. Statice pyramidata, *Prodr, MS.*

S. cespitosa, glauca: scapo erecto, ramoso, aphyllo: foliis radicalibus, parvis, obovato-oblongis, acutis, mucronulatis, in petiolum attenuatis, enerviis: paniculæ pyramidatæ ramis patentissimis, recurvis: floribus conglomerato-imbricatis; laciniis calycinis obtusiusculis.

Hab. in rupibus maritimis Ins^{æ}: Portûs S^{ti}.

Flores pallidè cærulei, parvi, glomerulis congesti. *S. auriculæfoliæ* et *oleæfoliæ* affinis. A *S. spathulatâ Desf.* differt foliis acutis; scapo magis ramoso; ramulis gracilioribus, sub-deflexis; floribus glomerulatis, minoribus, &c.

ORD. VIII. LABIATÆ.

GEN. SALVIA, *Linn., Spr.*

29. Salvia collina, *Prodr. MS.*

S. caule herbaceo, viscoso-piloso: calyce 5–dentato, $\frac{3}{2}$: foliis pinnatifidis, incisis, vel sub-sinuatis, dentatis, venosis, glabris, læviusculis: bracteis sub-rotundis, latis, cordatis, abbreviatis, acutis, inconspicuis, calycis dimidium æquantibus, integerrimis: verticillis 6–floris: corollis calycem duplò superantibus: galeâ falcatâ, compressâ: lobo medio labii inferioris cucullato; lobis lateralibus reflexis, parvis, abbreviatis, rotundatis, obtusis.

Salvia verbenacoides, *Brot?* (polymorpha, *Hoffm.*)

Hab. in collibus pascuisque altis Maderæ.

Salviæ pratensi et *Verbenacæ* (potissimùm priori) affinitas summa; sat verò distincta. A *Salviâ bicolori Desf.* differt caule humiliore; foliis glabris; bracteis abbreviatis, latis, acutis; lobis lateralibus labii inf^r^. rotundatis, &c.

GEN. THYMUS, *Linn.*

30. Thymus micans, *Sol. MSS.*

T. pedunculis ad apices ramulorum congestis, sub-racemosis, axillaribus, oppositis, solitariis, unifloris: calycis ½ labio superiore lato, obsoletè tridentato, marginibus recurvis; inferiore dentibus duobus suberectis, lanceolatis, acutis, contiguis, æqualibus profundè inciso: bracteis linearibus foliisque lineari-spathulatis, obtusis, basi attenuatis, pilisque raris, longis, patentissimis, remotis, pectinato-ciliatis: caulibus hispidis, prostratis, cespitosis, basi fruticulosis.

Thymus micans, *Herb. Banks.* et *Sol. MSS!*

Hab. copiosissimè, cespitem efficiens, per totum campum illum excelsum (5000 ad 6000 pedes altum) "Paul da Serra" dictum, Maderæ.

GEN. SATUREJA, *Linn.*

31. Satureja thymoides, *Sol. MSS.*

S. pedunculis axillaribus, multifloris, umbellatis: bracteis setaceis, fasciculatis: dentibus calycinis dimidium tubæ sub-æquantibus: foliis oblongo-lanceolatis, acutis, utrinque attenuatis, margine revolutis, subpuberulis, subtùs sub-incanis: ramulis junioribus sub-pubescentibus; caule fruticuloso, erecto.

Satureja thymoides, *Sol. MSS.* et *Herb. Banks!*

An Thymus therebinthaceus, *Willd. Enum. Pl. Hort. Berol. p.* 624?

Hab. in Maderâ et Portu S^ti^; vulgaris.

Fruticulus elegans. Folia sub-micantia, odorata. Flores sub-conspicui, purpurascenti-rosei. Stamina tubo corollæ breviora, inclusa,

Calyces striati, sub-pubescentes; fauce villis clausâ; dentibus setaceis, sub-æqualibus, duobus inf^{s}. aliquando sub-longioribus. Folia latitudine, &c. variabilia; ideoque *T. therebinthaceus Willd.* (si ex descriptione judicare licet), vix nisi genere discrepans, idem videtur.

Ord. IX. PERSONATÆ.

Gen. EUPHRASIA, *Linn., Spr.*

32. Euphrasia Holliana, *Prodr. MS.*

E. laciniis calycinis, foliisque lanceolato-oblongis, obtusis; inferioribus grossè dentatis; summis sub-integris: corollâ (luteâ) calycem duplò excedente; staminibus corollam sub-æquantibus: caule ramoso.

Bartsia viscosa, var. foliis linearibus obtusis &c. *Herb. Banks!*

Hab. in sylvis Maderæ.

Corollæ conspicuæ, sat magnæ; labium inferius 3-lobum, lobis rotundatis, obtusissimis, denticulatis; superius simplex. Folia et habitus quodammodò *Euphrasiæ Odontitis.*

Ob antheras distinctissimè aristatas, vera *Euphrasiæ* species: quinetiam toto habitu, caule ramoso &c. à *Bartsiâ viscosâ* differt. Ab *Euphrasiâ luteâ*, cui affinitate propior, dignoscitur caule ramisque robustis, nec filiformibus; foliis multò latioribus majoribusque, inferioribus grossè vel inciso-dentatis: floribus multò majoribus, &c.

Nomen dedi in honorem amici *Fr. Holl*, botanosophi Germanici, indefessi plantarum Maderensium indagatoris.

Gen. SCROFULARIA, *Linn., Spr.*

33. Scrofularia racemosa, *Prodr. MS.*

S. foliis sub-cordato-oblongis, acutis, sub-duplicato-serratis, utrinque cauleque acutangulo glabris, basi inæqualibus; inferioribus appendicu-

latis: thyrsi elongati aphylli ramis racemosis; racemis elongatis, flexuosis, patentibus, laxis, ramis pedicellisque sparsìm sub-glandulosis: calycis glabriusculi laciniis obtusis: corollæ labii inferioris lobo intermedio revoluto, vix prominulo, minuto; superioris, rudimento staminis 5ti. squamæformi, plano, rotundato.

Scrofularia auriculata. *Linn., Spr. &c.?*

α. longifolia; glaberrima, foliis acuminatis, elongatis, simplicitèr serrato-crenatis.

β. puberula; foliis radicalibus et junioribus subtùs petiolisque puberulis.

Hab. *α.* et *β.* ad rivulos et in rupibus madidis convallium Maderæ.

Scrof. sulphurea Mill. Dict.; et *S. Balbisii Wild.* (ex Spr.), saltèm ex descriptionibus, æquè plantæ nostræ *α.* pertinere possent. *S. auriculata* vera (cui prior forsan synonyma), foliis "obtusis subtus hirsutis" *Spr.* ("tomentosis" *Linn.*) "lobo terminali cordato aut ovato" *Desf.* differre videtur. *S. auriculatam Brot. Fl. Lusit. I. p.* 201 vero, ob folia "subtus glabra," ab *â.* nostrà alienam ægrè putarem, nisi quòd à nonnullis ad *S. trifoliatam Linn.* relatam video. In re tam dubiâ, plantam Maderensem pro tempore distinctam servandam putavi.

34. Scrofularia hirta, *Prodr. MS.*

S. foliis cordato-oblongis, acutiusculis, basi sub-æqualibus, excisis, argutè duplicato-inciso-serratis, utrinque petiolis cauleque acutangulo villosis; petiolis latis, sub-alatis, ex-appendiculatis, hirtis; thyrsi aphylli ramis trichotomo-racemosis; racemis elongatis, pedunculisque glanduloso-pubescentibus; calycis glaberrimi laciniis obtusissimis; corollæ labii inferioris lobo intermedio revoluto, vix prominulo, minuto; superioris, rudimento staminis 5ti. papillæformi, minutissimo, brevi; genitalibus exsertis.

Hab. in Maderæ umbrosis humidis obscuris. Rariss.

ORD. X. CONVOLVULEÆ.

GEN. CONVOLVULUS?

35. Convolvulus? solanifolius, *Prodr. MS.*

C? caule volubili, fruticoso: foliis cordatis, ovato-oblongis, acutis, integerrimis, petiolatis; junioribus, ramulis, petiolisque pubescentibus: pedunculis axillaribus, solitariis, petiolo longioribus, apice sub-trifloris, pedicellisque elongatis, nudis: calycibus ovalibus, obtusiusculis; . . . . .

Hab. in Maderæ rupibus. Rariss.

Corollam nondum vidi.

ORD. XI. SAPOTEÆ.

GEN. SIDEROXYLON, *Spr.*

36. Sideroxylon Mermulana, *Prodr. MS.* "*Mermulana*," incolarum.

S. inerme: foliis obovatis, obtusis, spathulatis, integerrimis, coriaceis, nervosis, lucidis, utrinque glaberrimis: pedunculis unifloris, ad axillas aggregatis, brevibus, calycibusque velutinis.

Sideroxylon Mermulano, *Herb. Banks.!*

Hab. in rupibus, præsertim maritimis, Maderæ.

Frutex, vel sub-arboreum. Flores parvi, pallidè carnei. Fructus ruber, edulis.

Nomen "*Mermulana*" in Canariis plantæ distinctissimæ, sed quoad habitum simillimæ, sc. *Myrsinei canarienſi*, impositum scribit amicus P. B. Webb, arm.

ORD. XII. COMPOSITÆ.

* CICHORACEÆ.

GEN. SONCHUS, *Linn., Spr.*

37. Sonchus ustulatus, *Prodr. MS.*

S. glaberrimus: caulibus simplicibus, brevissimis, herbaceis, basi sub-lignosis: foliis radiatim confertis, decursivè runcinato-pinnatis, sub-

carnosis, rigidis, subtus præsertim inter venas pulchrè glaucescentibus; foliolis acutis, angulatis, sub-integerrimis vel dentibus sparsis, raris, minutis, callosis: scapi terminalis, aphylli, pauciflori, ramis raris, divaricatis, solidis, pedunculisque 1-floris, suprà incrassatis, paniculatis, nudis; squamis anthodii purpureo-nigricantibus, adpressissimis, latis.

S. hyoserifolius, (Hornem:) *Spr?*

α. ***angustifolia;*** foliolis angustis, confertis, acuminatis, margine posteriore sub-integerrimo.

Sonchus dentatus, ***Herb. Banks?***

β. ***latifolia:*** foliolis majoribus, latioribus, distantibus, angulatis, utrinque denticulatis: foliis suprà vix glaucescentibus, profundiùs incisis.

"Sonchus squarrosus β" (lineâ atramenti per medium verborum ***squarrosus*** β ductà, et "fruticosus" suprà scripto) "et MSS. differt paniculâ dichotomâ—planta minor. Madera Fr. Masson." ***Herb. Banks!***

Hab. in rupibus maritimis aridis Maderæ.

β (vix var.) est status potiùs è solo vel humidiore vel magis umbroso ortus.

De *Soncho hyoseridifolio* huc ritè referendo, suspensus hæreo. Characteres ferè iidem; nisi quòd illa inter "***fruticosos***" numeratur, cùm nostra planta certissimè "***herbaceis***" releganda est. An descriptio cl. Sprengelii à specimine manco, (sc. sine radice desiccato), in herbario servato, (quale est forsan ***Sonchus dentatus Herb. Banks.***), factitata; ideoque erronea? Nam in tali, caulis casu forsan quodam lignosus evadere posset: specimine tùm speciem ramuli fruticis cujusdam ramosæ omninò præbente.

S. dentatus Herb. Banks. in omnibus nisi caule lignoso cum varietate α. nostrâ convenire videtur. Sed, cùm præter hoc, alia exstant specimina in ***Herb. Banks.***, ad β. nostram certissimè pertinentia, quæ à cl. Solandro ad alteram (quamvis revera distinctissimam et longè alienam) speciem (*S. squarrosum*) referuntur, ideoque à *S.* suo *dentato* planè distinguuntur, impensiùs suspicandum est hunc *S. dentatum* à

nostrâ plantâ alienum esse, et forsan *S. hyoseridifolii* (Hornem.) *Spr.* veri synonymam. In re tam incertâ, difficultatem minùs nomine planè novo quàm veteri incerto augeri putavi.

Gen. TOLPIS, *Gaert.*

38. Tolpis crinita, *Prodr. MS.*——*Tab.* II. ff. 1—3.

T. caule ramoso: ramis virgatis: foliis radicalibus humifusis, solo adpressis, plerumque sinuato-pinnatifidis, sub-canescentibus; caulinis angustis, lanceolato-linearibus: bracteis setaceis, abbreviatis, ad apices pedunculorum infernè nudorum congestis: seminibus (omnibus, radii sc. conformibus) sub-quadrisetis.

Crepis crinita, *Sol. MSS. et Herb. Banks!*

Crepis incrassata, *Herb. Banks.* "Insulæ Azores Fayal Mess". Forster"!

Hab. in Maderæ collibus apricis.

Gen. CREPIS, *Spr.*

39. Crepis pectinata, *Prodr. MS.*

C. caule frutescente, ramoso, foliato; ramis diffusis, virgatis: foliis flaccidis, tenuissimè et profundè divisis, pectinato-pinnatifidis; laciniis distantibus, elongatis, lineari-filiformibus, suprà glabris, subtùs subfarinosis: pedunculis proliferis, supernè incrassatis, squamosis, squamisque minutis, erectis, anthodioque farinoso-albescentibus.

Crepis tenuifolia, *Sol. MSS. et. Herb. Banks!* non *Willd.*

Hieracium fruticosum foliis tenuissimè coronopi modo divisis. *Sloan. Cat.* 123.—*Hist. Jam. p.* 19. *t.* 5. *f.* 1, 2. (Icon. mala; Descr. opt.)

Hab. in rupibus apricis Maderæ ubique.

Affinitate et habitu *Crepidi succulentæ Hort. Kew:* (*C. coronopifoliæ Desf:*) proxima; cujus speciei, præ ceteris polymorphæ, (in Maderâ vix minùs vulgaris), varietatem esse meram alia forsan dies docebit. In illâ tamen, quamvis folia aliquando profundiùs pinnatifida

quàm in statu normali, nunquam ut in *C. pectinatâ* nostrâ ferè filiformia et tenuitèr divisa, laciniis elongatis linearibus (ferè ut in *Coreopside tinctoriâ Hort.*) videntur.

Obs. *C. succulenta Ait.* et *C. pectinata nob:* genus *Tolpidem* cum *Crepide* arctissimè conjungunt.

40. Crepis macrorrhiza, *Prodr. MS.*

C. glaberrima: radice perenni, crassâ, carnosâ: caulibus solidis, foliatis, simplicibus, supernè paniculatis: foliis omnibus indivisis, oblongis, dentatis, sessilibus, nitidis, sub-carnoso-coriaceis: panicula largâ, multiflorâ; pedunculis supernè sub-incrassatis, squamosis; anthodiis subfarinoso-pubescentibus.

Crepis macrorrhiza, *Herb. Banks: et Sol. MSS! Hook. in Bot. Mag. t.* 2988*!*

Hab. in Maderæ rupibus.

41. Crepis? andryaloides, *Prodr. MS.*

C? glanduloso-hispida: radice carnosa, bienni: caule sub-fistuloso, foliato, simplici, supernè laxè paniculato, hispido: foliis omnibus indivisis, oblongis, acuminatis, undulatis, remotè runcinato-dentatis, subsinuatis, sessilibus, hispidis: floribus laxè paniculatis, remotis: pedunculis nudis, gracilibus, divaricatis, laxis, anthodiisque cylindricis, glanduloso-hirsutissimis: involucro erecto, persistente.

Hab. in convallibus Maderæ.

Semina matura non vidi; pappus in immaturo reverà sessilis: sed cum in veris quibusdam *Borkhausiæ* speciebus pappus in semine immaturo omninò *sessilem* vel *subsessilem* videre licet, in nostrâ forsan plantâ pappus seminis maturi stipitatus evadit.

Gen. BORKHAUSIA, *Böhm. Spr.*

42. Borkhausia laciniata, *Prodr. MS.*

B. radice annuâ: caule erecto, stricto, ramoso, paniculato, subpuberulo, nitido: foliis laciniato-pinnatifidis, vel runcinato-dentatis, sinu-

atis, glabris; radicalibus plerumque integriusculis, oblongis; caulinis lineari-lanceolatis, semi-amplexicaulibus, basi auriculatis, sub-sagittatis, dentato-laciniatis: floribus corymboso-paniculatis: anthodii squamis dorso infernè nigrescenti-glanduloso-hispidis, interstitiis sub-farinoso-puberulis: squamis involucri laxis, farinoso-puberulis.

α. *pinnatifida;* foliis profundiùs divisis.

Crepis biennis, *Herb. Banks!* quoad specimina in Maderâ a Masson lecta.

"Crepis Dioscoridis" (lineâ per verbum *Dioscoridis* ductâ) "L. var. corolla undique lutea. Madeira Fr. Masson 1777." *Herb. Banks!*

β. *integrifolia;* foliis integriusculis.

"Crepis Dioscoridis" (lineâ per verbum *Dioscoridis* ductâ) "L. var. foliis margine nudis, Madeira Fr. Masson 1777." *Herb. Banks!*

Hab. in Maderâ; in vinetis, locis cultis, frequens.

43. Borkhausia divaricata, *Prodr. MS.*

B. radice crassâ, fusiformi, bienni (perenni?): caulibus ramosis, paniculatis, solidis, infernè glabris, supernè pedunculisque divaricatis, patentibus, hispido-glandulosis: foliis rigidis, glaberrimis, undulatis; radicalibus sinuato-runcinatis, caulinis basi semi-amplexicaulibus, dilatatis, ovato-acuminatis, integriusculis: floribus sparsis, paniculatis: anthodiis post anthesin ovatis, basi ventricosis; squamis basi hispido-glandulosis, supernè squamisque laxiusculis involucri glabris.

α. *robusta;* caulibus erectis, virgatis, pedalibus, multifloris, foliosis: foliis sæpe runcinato-pinnatifidis.

Hab. in Promontorio S^ti^. Laurentii Maderæ.

β. *pumila;* caulibus sæpe diffusis, glabriusculis, paucifloris, plerumque nudis: foliis radicalibus indivisis, integriusculis vel runcinato-sinuato-dentatis, sub-carnosis: anthodiis hispidioribus.

Hab. in Portû S^to^.—Status potiùs, ex solo aridiore, quàm varietas præcedentis.

44. Borkhausia hieracioides, *Prodr. MS.*

B. radice annua?: caule erecto, ramoso, paniculato, foliarum costâ centrali, pedunculis, anthodiisque setoso-hispidis vel sub-muricato-spinellosis: foliis glabris, indivisis, denticulatis, denticulis raris, sparsis, minutis, subulatis; radicalibus lanceolato-oblongis, acutis, basi attenuatis; caulinis ovato-acuminatis, basi dilatato-auriculatis, semi-amplexicaulibus: floribus corymbosis; flosculorum ligulis elongatis, laxis, patentissimis, sub-pendulis.

a. integrifolia; foliis integriusculis, sub-sinuolatis.

β. pinnatifida; foliis sub-pinnato-runcinatis.

Hab. in Maderæ orâ Septentrionali.

45. Borkhausia dubia, *Prodr. MS.*

B. radice bienni: caule erecto, stricto, è basi ramoso, ramisque foliatis, costaque centrali foliarum subtùs hispidis: foliis lucidis, glabris, indivisis, marginibus undulatis, sinuato-runcinatis et denticulatis, denticulis intermediis plurimis, inæqualibus, subulatis vel ciliato-setaceis; radicalibus elongatis, oblongo-lanceolatis, acutis, basi attenuatis; caulinis basi cordato-æqualibus, amplexicaulibus, oblongis, acuminatis; summis linearibus, sub-integerrimis, setaceo-ciliatis: floribus sub-corymbosis: ramulis supernè, pedunculis, anthodiisque densè glanduloso-pubescentibus, sub-incanis, farinaceo-puberulis: pappo sub-stipitato.

Hab. in convallibus Maderæ.

Præcedenti proxima; differt autem habitu distinctissimo, ramis supernè, pedunculis, anthodiisque densè glanduloso-pubescentibus, sub-incanis, farinaceo-tomentosis; floribus minoribus in corymbos laterales collectis, flosculorum ligulâ nec elongatâ nec pendulâ, pappo sub-stipitato (quo ad *Crepides* veras accedit), foliarum margine inæqualiter sed conspicuè et in omni parte runcinato-sinuatis, caulinis basi æqualibus, cordatis, (non dilatato-auriculatis).

46. Borkhausia comata, *Prodr. MS.*

B. radice fusiformi, carnoso: caule erecto, è basi ramoso, foliato, hirto-setoso: foliis indivisis, denticulatis; radicalibus glabris; caulinis

summis ciliato-crinitis: floribus corymbosis: anthodiis hirsutissimis, comatis; squamis crinitis.

Crepis comata, *Herb. Banks. et Sol. MSS!*

"Hab. in Maderæ sylvis; Fr. Masson 1777." *Sol.*

Pappus distinctissimè stipitatus.

Gen. THRINCIA, *Roth, Spr.*

47. Thrincia nudicaulis, *Prodr. MS.*

T. foliis hispidis, sub-dentato-sinuatis: pappo disci stipitato.

Leontodon nudicaule, *Herb. Banks!*

Hab. in apricis Maderæ ubique; vulgatissima.

Flosculorum tubulus ad apicem pilosus; ligulorum laciniæ eglandulosæ. Pappus radii paleaceus; disci plumosus. Semina sursùm attenuata, acuminata, in rostrum gracile, elongatum producta; unde pappus stipitatus.

Thrincia hirta Hook. Brit. Fl. (*Apargia hirta Sm. Eng. Fl., Hedypnois hirta Ejusd.* in *Engl. Bot.*) pappo disci sessili, potissimùm differt. Eodem charactere, necnon genere, sc. pappo radii paleaceo, ab *Apargiâ hispida* omninò distincta.

* * CINAROCEPHALÆ

Gen. CIRSIUM, *Tourn, Spr., &c.*

(CNICUS, *Aliorum*).

48. Cirsium latifolium, *Prodr. MS.*

C. inerme: foliis sessilibus, basi auriculatis, amplexicaulibus, omnibus elliptico-oblongis, latis, obtusis, indivisis, latè sinuato-crenatis, setoso-spinelloso-ciliatis, supra lucidis, nudis, subtus cauleque lanato-tomentosis, floccosis: pedunculis longissimis, floccosis, unifloris: anthodiis sub-lanatis: squamis lanato-ciliatis, mucronatis, adpressis, inferioribus ovatis, acutis; superioribus oblongis, obtusiusculis.

Carduus latifolius, *Herb. Banks!*

Hab. in Maderæ convallibus.

Species pulchra, distinctissima, C. *heterophyllo* affinis.

Caulis 2—3-pedalis. Folia ampla, subtùs sæpe nivea. Flores purpurei.

* * * CORYMBIFERÆ.

GEN. GNAPHALIUM, *Linn., Spr. &c*

49. Gnaphalium melanophthalmum, *Prodr. MS.*

G. fruticosum: foliis sparsis, sessilibus, lanceolatis, acuminatis, basi attenuatis, ramisque niveo-tomentosis, canescentibus: paniculis terminalibus, congestis, corymbosis: squamis anthodii nivei, globosi, laxis, ovatis; inferioribus obtusis, rotundatis; superioribus acutiusculis.

Gnaphalium rupestre, *Herb. Banks!*

OBS. *Gnaph. rupestre, Rafin:* jam adest. *Steud. Nom. Bot.*

Hab. in rupibus convallium Maderæ.

Flores nivei, odori; disco post anthesin nigro.

ORD. XIII. RUBIACEÆ.

GEN. GALIUM, *Linn., Spr.*

50. Galium productum, *Prodr. MS.*

G. glabrum: foliis octonis, lanceolato-linearibus, acutis, cuspidatis, reflexis, sub-integerrimis, denticulis marginalibus raris, obsoletis, antrorsùm spectantibus, utrinque lævibus, suprà cauleque parùm ramoso lucidis: panicularum lateralium terminaliumque ramis divaricatis, abbreviatis: corollæ laciniis obtusiusculis, mucronatis: fructibus lævibus, glabris: caule 4-angulari, debili, diffuso, elongato, simpliciusculo, lævi, basi suffruticoso.

Hab. in Maderæ saxosis, sepibus, rupibus &c. frequens.

Ord. XIV. UMBELLIFERÆ.

Gen. ŒNANTHE, *Spr.*

51. Œnanthe pteridifolia, *Prodr. MS.*

S. radicibus tuberosis, fusiformibus, fasciculatis: caule erecto, infernè tereti, lævi, ramis angulatis, striatis: foliis omnibus tripinnatis; pinnis pinnulisque omnibus remotis, oppositis, patentissimis, distichis; foliolis ultimis ovatis lanceolatisque, acutis, inciso-dentatis pinnatifidisque, basi cuneatis: umbellis oppositifoliis; radiis inæqualibus; bracteis paucis, subnullis, bracteolisque linearibus: fructu suberoso.

Œ. apiifolia, *Brot?*

Hab. in rupibus madidissimis convallium Maderæ.

Radices repentes; tuberibus fusiformibus, fasciculato-filipendulis, crassitie digiti. *Caules* elati, fistulosi, esculenti. *Folia* maxima, elegantia, lætevirentia, foliolis exiguis, tenuibus, concinnis. *Umbellæ* mediocres, sat parvæ; floribus albidis, aspectu eorum *Ænanthes crocatæ*. *Calyx* persistens. *Petala* mucrone elongato, inflexo. "*Floral Receptacle*," *Sm.* nullum. *Stylopodia* ("*Bases of Styles*," *Sm.*) tumida, globosa. *Styli* persistentes, post anthesin elongati, fructum maturum æquantes. Fructus ovato-oblongus, lateralitèr (sc. suturâ) compressus, præsuberosus. *Mericarpia**, striis 7 dorsalibus, lævibus, sub-æqualibus, tribus vix majoribus; interstitiis angustis, planis, æquis; sutura utrinque spatio tumidulo, latiusculo, lævi, spongioso vel suberoso. *Albumen* sive *Perispermium* teretiusculum, intùs (plano-convexum), *Vittæ* 6, rectæ, æquales; 4 dorsales, æquidistantes; jugis tribus dorsalibus sub-majoribus alternantes, sc. 4 intermediis oppositæ; reliquæ duæ

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* Hæc forsan melius ita describenda: *mericarpia jugis* 5, tribus dorsalibus filiformibus, sub-prominulis; duobus lateralibus marginantibus dilatatis, spongiosis, spatium latum, convexum, tumidulum, suberosum utrinque formantibus; *valleculis* 1-vittatis, 1-striatis; striis filiformibus, distinctis, juga subæquantibus, sc. vix minùs prominulis; hinc mericarpia 7-striata apparent. *Carpophorum* evanidum, sub-nullum.

juncturam respicientes, approximatæ. Totius plantæ succus aquosus. An Genus?

Gen. SAMBUCUS, *Linn., Spr.*

52. Sambucus nigra, *Linn. Sm., &c.*

α. communis; foliolis ovatis.

Sambucus nigra, *Auct.*

Hab. in Europâ; Angliâ, &c.

β. lanceolata: foliolis lanceolatis vel ellipticis, elongatis.

Sambucus lanceolata, *Herb. Banks.*

Hab. in Maderæ sylvis: in hortis etiam ab incolis colitur.

Præter foliola magis elongata, omnia ut in *α*; ideoque vix species consenda.

Ord. XV. CRASSULACEÆ.

Gen. SEDUM, D.C.

53. Sedum fusiforme, *Prodr. MS.*—Tab. 3. ff. 1, 2.

S. caule fruticuloso, ramoso; ramulis confertis, erectis, tortuosis, glabris, infernè nudis: foliis omnibus sparsis laxis sub-patentibus, carnosis, crassis, fusiformibus, sub-teretibus, suprà planiusculis, acutiusculis, utrinque attenuatis, glaberrimis, glaucis: cymis terminalibus, cormboso-fastigiatis, paucifloris: petalis 5, lanceolatis, obtusiusculis, patulis: squamis nectariferis brevibus, lunatis.

Hab. in Maderæ rupibus excelsis aridis maritimis.

Ramosissima, cespitosa, humilis. Flores flavi.

S. altissimo proxima; habitu prorsùs *S. nudi*, cui maximè affinis.

54. Sedum farinosum, *Prodr. MS.*

S. candicans: caulibus herbaceis, prostratis (repentibus?), elongatis, infernè nudis, sub-simplicibus; foliis ad apices confertis, 4-fariis,

caulibusque albo-farinosis, teretibus, supra sub-planulatis, obtusissimis: cymis terminalibus, 3–partitis: petalis 5, ovato-lanceolatis, acutis; squamis nectariferis . . . . . carpellis rostratis, acutis.

Hab. in rupibus umbrosis Maderæ, ad altitudinem 4000 ad 5500 ferè pedum.

Petala alba, nervo extrà rubro.

Ord. XVI. LYTHRARIEÆ.

Gen. LYTHRUM, D.C.

55. Lythrum junceum, *Sol. MSS.*

L. floribus axillaribus, hexapetalis, dodecandris: filamentis 6 brevissimis; 6 longioribus, tubo brevioribus; antheris sub-inclusis: calycis angulati dentibus alternis minoribus: foliis alternis, confertis, lanceolato-linearibus, sub-glaucescentibus: caulibus acutè 4–angularibus, debilibus, humifusis, elongatis; deorsùm nudis, suffrutescentibus.

Lythrum junceum, *Sol. MSS! et Herb. Banks.* quoad specimina Maderensia!

——— acutangulum, *Lagasca*, (L. Græfferi var? *D. C.*)?

Hab. in Maderæ humidis, frequens.

Caules graciles, demùm prælongi, simpliciusculi, infernè nudi, frutescentes. Folia parva, sub-conferta. Flores hexapetali, magnitudine mediocri, lætè purpurei.

Ord. XVII. ROSACEÆ.

Gen. RUBUS, *Linn., D. C.*

56. Rubus grandifolius, *Prodr. MS.*

R. caulibus fruticosis, angulatis, aculeatis, glabris, procumbentibus, sterilibus elongatis; aculeis sparsis, compressis, recurvis, numerosis: foliis quinatis (rarò ternatis), sub-pedatis; foliolis ovato-oblongis, acumi-

natis, grossè duplicato-serratis, utrinque glaberrimis, nudis, longè petiolulatis; petiolis petiolulisque sparsìm aculeatis: paniculæ elongatæ, terminalis, ramis pedunculis calycibusque densè purpureo-glandulosis: laciniis calycinis reflexis, inermibus, petalis multò brevioribus.

Rubus pedatus *Herb. Banks! et Sol. MSS!* non *Smith.*

Hab. in rupibus Maderæ.

Folia lucida, utrinque viridia, magna. Flores albi, conspicui, magni. Fructus sat magni, atri.

Ord. XVIII. LEGUMINOSÆ.

Gen. VICIA, *Tourn. D. C.*

57. Vicia albicans, *Prodr. MS.*

V. annua, villosa, sub-canescens: caulibus tetragonis: cirris valde ramosis: foliolis oblongis, mucronatis, numerosis, oppositis et alternis: stipulis semi-sagittatis, inciso-dentatis: pedunculis sub-bifloris, folio multùm brevioribus; floribus secundis, laxis, sub-remotis: dentibus calycinis duobus superioribus minimis, obsoletis; inferioribus ovato-subulatis, medio longiore; omnibus tubo brevioribus et cum toto calyce colorato pilosis: stylis capitatis, infra capitulum globosum undique, subtùs verò præsertim, barbatis: leguminibus oblongis, latiusculis, brevibus, sub-compressis, albido-hirsutissimis, pendulis, sub-tetraspermis; seminibus globosis, viridi-fuscis, nigro-maculatis, glabris.

Hab. in rupestribus aridis apricis Maderæ.

Flores magnitudine mediocri, paullò sc. majores quàm in *V. Craccâ*, rosei vel purpurei, apice purpureo-nigro, vexillo striato. *V. atropurpureæ, Desf.; trichocalyci, Moris.; Broterianæ, Ser.* in *D. C. Prodr.* (*V. villosa Brot* non *Roth.*) affinis. Radice annuâ à *V. perenni D. C.; argenteâ, Lapeyr.; variegatâ Willd.; alpestri Stev.; cinereâ Bieb.*, necnon aliis notis distincta.

58. Vicia micrantha, *Prodr. MS.*

V. annua, gracilis, glabriuscula: caulibus filiformibus: cirris ramosis: foliolis angusto-lanceolatis vel lineari-oblongis, remotiusculis,

ob-tusiusculis, sub-puberulis: stipulis parvis, angustissimis, semi-sagittatis, superioribus simplicibus: pedunculis sub-bifloris, folio multùm brevioribus: calyce leguminibusque latis, oblongis, compressis, 3—6-spermis, villosis.

Vicia gracilis, *Sol. MSS. et Herb. Banks!* non *Loisl.*

Hab. in Maderâ; *Sol. et Mass.*

Foliola sub-octoparia. Flores perparvi, purpurascentes.

GEN. ONONIS, *Linn., D. C.*

59. Ononis dentata, *Sol. MSS.*—Tab. 4.

O. herbacea, annuâ, erecta, pilosa: foliis (omnibus) trifoliolatis; foliolis obovatis, serratis: stipulis ovatis, dentatis: floribus sparsis, solitariis, axillaribus, pedunculatis, folio longioribus, cernuis: pedunculis muticis: corollâ calycem superante: laciniis calycinis 4 supremis anticè dilatatis, foliaceis (3–) dentatis; infimâ simplici lineari-acuminatâ, integerrimâ; leguminibus calyce longioribus.

Ononis dentata, *Sol. MSS! et Herb. Banks.* quoad specimina 3 in Insulis Canariis A. D. 1778 à cl. Masson lecta!

Hab. in Portu S^to^. "Insulæ Canariæ Fr. Masson 1778," *Herb. Banks:* In apricis Nivariæ, P. B. Webb, arm.

Flores conspicui; in plantis ab amico Rev.° M. J. Berkeley in Angliâ cultis (à seminibus quæ in Insulâ Portûs S^ti^. mense Maii, A. D 1828, ipse legi) vexillo roseo-purpureo, alis et carinâ pallidioribus; in aliis (desiccatis) ab amico P. B. Webb arm. in Nivariâ lectis, pallidè flavi, carinâ purpureâ.

GEN. ASTRAGALUS, *D. C.*

SERIES II. *OCHROLEUCI,*

§. 7. Bucerates, *D. C. Prodr.*

60. Astragalus canescens, *Sol. MSS.*

A. villoso-pubescens, adscendens: caulibus diffusis, adscendentibus: foliolis multijugis, ovalibus vel oblongo-ellipticis, retusiusculis, suprà

glabris, infrà hirtis, canescentibus: pedunculis elongatis, folio multùm longioribus: racemis multifloris: pedicellis fructiferis deflexis: leguminibus falcatis, compressis, dorso canaliculatis, apice acutis, pubescentibus, pendulis; sulci dorsalis lati, profundi marginibus acutis.

Astragalus canescens, *Sol. MSS. et Herb. Banks!*

Hab. in Insulâ Portu S^to^. etiam Canariis ab amico P. B. Webb arm. lectus.

Flores pallidè flavi, virescentes. *A. hamoso* proxima; nec forsan verè distincta.

Ord. XIX. HYPERICINEÆ.

Gen. HYPERICUM, *D. C.*

61. Hypericum angustifolium, *Prodr. MS.*

H. glabrum: caulibus simplicibus erectis, strictis, virgatis, ancipitibus, suffrutescentibus: foliis epunctatis, erectis, lineari-oblongis, obtusissimis vel retusis, amplexicaulibus, margine revolutis: paniculâ terminali, corymbosâ: sepalis ovatis, æqualibus, dentato-glandulosis petalisque nigro-punctatis: floribus trigynis . . . . . .

Hab. in Maderæ campo præcelso (5000—6000 ped. alt.) "Paul da Serra" dicto.

Caules plures, ferè pedales, tenues.

Ord. XX. MALVACEÆ.

Gen. SIDA, *Cav., D. C.*

Sect. *MALVINDA, Med., D. C.*

* * *Oblongifoliæ;* nempe pedicellis elongatis, distinctiùs articulatis, foliis oblongis ovatisve. *D. C. Prodr.*

62. Sida maderensis, *Prodr. MS.*

S. fruticulosa: foliis lanceolatis oblongisve, acutis, serratis, glabris, subtùs pallidis, sub-glaucis, breviter petiolatis: axillis inermibus: pedicellis axillaribus, unifloris, inæqualibus, folio brevioribus: carpellis 10—12, uni-rostratis.

Malvinda unicornis folio rhomboide perennis, *Dillen. Hort. Elth. p.* 216, *t.* 172. *f.* 212. (descr. et fig. opt.)

Hab. secùs vias in locis incultis &c. Maderæ; in regione totâ inferiore vulgatiss:

Pedicellis nunquam "folii longitudine" et carpellis pluribus, semper uni-rostratis, à *S. canariensi* differt. Flores parvi, ochracei. Fruticulus.

Obd. XXI. VIOLARIEÆ.

Gen. VIOLA, *Tourn., D. C.*

Sect. I. *Nomimium, Ging.*—

§. 2.——*, *D. C. Prodr.* I. *p.* 295.

63. Viola maderensis, *Prodr. MS.*

V. caulescens, stolonifera: caulibus brevibus, erectis, suffrutescentibus, glabris: foliis profundè cordatis, rotundato-ovatis, sub-pubescentibus; petiolis elongatis pedunculisque pube deflexâ hirsutis; stipulis glabris, acuminatis, glandulis ciliato-serrulatis: sepalis oblongis, acutis: petalis lateralibus vix sub-barbatis: calcare sub-compresso, saccato, plerumque obtusissimo, (rarò acuminato): stigmatis rostro uncinato, deorsùm (sc. ad flexuram) immarginato, nudo, complanato (nec convexiusculo), styloque compresso, simplici, glabro: capsulis pubescentibus, hexagonis, globosis, abbreviatis: seminibus albidis, pallidè flavescentibus, obovatis.

Hab. in Maderæ sylvis, ubique vulgatissima.

Flores odoratissimi, violacei, sub-pallidiores quàm in *V. odoratâ*.

Ord. XXII. CRUCIFERÆ.

Gen. SINAPIDENDRON, *nob.*: *Prodr. MS.*

Sinapis, *Sect.?* 5ta *Disaccium, D. C.*

Sinapis, *Brown*, in *Hort. Kew., Hook.*

Hesperis, *Spr.*

Calyx clausus, demùm erecto-patens; basi sub-bisaccatus. *Stylus* distinctus. *Stigma* capitatum. *Siliqua* linearis, teretius-

cula, sub-torulosa, flexuoso, rostrata, basi tetragona; *septo* subspongioso. *Semina* uniseriata, oblonga. *Cotyledones* incumbentes, sub-conduplicatæ.

Suffrutices Maderenses. Folia sub-carnosa, rigida, simplicia. Flores flavi, inodori. Siliquæ graciles, elongatæ, pedicellatæ. Genus habitu, calyce, siliquis, seminibusque à Sinapi distinctum.

SPECIES.

64. Sinapidendron frutescens, *Prodr. MS.*

Sinapis frutescens, *Ait: Hort. Kew:* IV. *p.* 127. *n°.* 11.—*Herb. Banks! D. C. Prodr.* I. *p.* 220. *n°.* 34.—*Hook: Misc. Bot.* I. *p.* 119. *t.* 28*!*

Hesperis diffusa, *Spr. Syst.* II. *p.* 900. *n°.* 18.

Hab. in rupibus Maderæ.

65. Sinapidendron salicifolium, *Prodr. MS.*

S. "caule frutescente; foliis lineari-lanceolatis, integerrimis." *Sol. MSS.*

Brassica frutescens, *Sol. MSS. et Herb. Banks!*

"Hab. in Maderâ inter rupes maritimas prope vicum Camara de Lobos. ♃. *Fr. Masson.*" *Sol. MSS.*

Species videtur à *S. frutescente* distincta. *Folia* succulenta, conferta, integerrima, sub-obtusa, 2—3 poll. longa, $\frac{1}{4}$ poll. lata. *Calyx* semipatens. *Siliquæ* 1—1$\frac{1}{2}$ poll. (absque rostro) longæ, lineares, flexuosæ, graciles, 4-angulares, longitudinalitèr sub-striatæ; rostro $\frac{1}{3}$—$\frac{1}{4}$ poll. longo, capitato, subulato, sub-compresso coronatæ.

Habitus omninò *S. frutescentis.* Plantam vivam nondum vidi: descriptio à specimine Banksiano composita est.

66. Sinapidendron rupestre, *Prodr. MS.*

S. caule basi frutescente petiolis, foliisque crassiusculis, strigoso-hispidis; superioribus elongatis, oblongo-linearibus, integerrimis; inferioribus ovato-oblongis, sinuato-dentatis, basi sub-lyratis, petiolatis: siliquis glabris; rostro ancipiti, brevi.

α. *chætocalyx;* pedicellis, calycibus maculatis, germinibusque hispidis.

Hab. in rupibus convallium Maderæ.

β. *gymnocalyx;* siliquis sub-abbreviatis, pedicellis, calycibus sub-immaculatis, germinibusque glabris: foliis lucidis; inferioribus rotundatis obtusis, setis raris scabris.

Hab. in rupe quâdam excelsâ maritimâ, ad locum oræ Septentrionalis Maderæ "Entroza" dictum: semel tantùm legi.

In α, Calyx purpureo-nigro maculatus. Flores majusculi; petalorum limbo citrino; ungue purpureo.

β. An species? sed habitu eodem gaudet; nec in ceteris characteribus, floribus &c. præter suprà indicata, differt.

Ord. XXIII. RANUNCULACEÆ, *D. C.*

Gen. RANUNCULUS, *C. Bauh., D. C.*

§§. II. *Ranunculastrum, D. C.*

67. Ranunculus grandifolius, *Prodr. MS.*

R. foliis amplissimis, lucidis, cauleque hirsutiusculis; radicalibus petiolatis, orbiculato-reniformibus, latis, sub-quinquelobis, dentatis; lobis abbreviatis, rotundatis: caule elato, ramoso, corymboso; ramis divaricatis, sub-patentibus: corymbo vasto, amplo: calyce patentissimo.

Hab. in rupibus humidis umbrosis Maderæ; præsertim Convallis frigidæ (*Ribeiro Frio* dictæ).

Characteribus difficillimè, habitu sc. staturâ, toto cœlo ab affinibus *R. cretico* et *R. cortusæfolio* dignoscitur. Plantam in horto cultam nec solo, nec cœlo aridiore mutatam inveni. Folia radicalia sub-indivisa, diametro ferè pedali. *Caulis* 2—3-pedalis. *Flores* conspicui, flavi, magni; *petalis* sc. 1 poll. longis.

MOLLUSCA.

CLASS: GASTEROPODA.

ORD. PULMONEA.

1. *Familia, Limacidæ.*

I. GENUS ARION, *Fer.*

1. ARION empiricorum, *Fer.*

   a. Varietatis *a. Fer.* sub-varietates duæ; altera olivacea vel fusco-lutescens; altera pallidior, cæruleo-cinerascens.

   δ. Fer.

   Hab. in Maderâ.

II. GENUS, LIMAX. *Fer.*

2. Limax antiquorum. *Fer.*

   a. Fer. sub-varietates.

   η. Fer.?

   Hab. in Maderâ.

3. Limax variegatus, *β. Fer.*
   Hab. in Maderâ.

4. Limax agrestis, *Fer.*

   ε. Fer.

   η. Fer.

   Hab. in Maderâ.

III. Genus, TESTACELLUS, *Cuv.*

5. Testacellus haliotideus, *Drap., Sow., Fer.*
   Hab. in Maderâ.
6. Testacellus Maugei, *Fer., Sow.*
   Hab. in Maderâ.

2. *Familia, Helicidæ.*

IV. Genus, VITRINA, *Drap.*

Helicolimax, *Fer.*

7. Vitrina Lamarckii, nob. in *Zool. Journ.*—Tab. 5. ff. 1, a, b.
   Helicolimax Lamarckii, *Fer.*
   Hab. in Maderâ et Portu S^to^.

V. Genus, HELIX, *Fer.* (excluso Sub-genere *Cochlodina*, i. e. *Clausilia*).

Obs. Methodum cl. Ferussaci hinc usque ad finem Cochlodontium sequor.

§§. Inclusæ.

†. Volutatæ, *Helicoides.*

I. Sub-genus. Helicogena.

1. *Columellatæ;* columella solida torta; globosæ.

8. Helix furva, *Prodr. MS.*—Tab. 5. f. 2.

H. testa imperforata, sub-globosa, tenui, fusco 1-fasciata; epidermide umbrino: anfractibus obsoletè rugulosis, primo carinato, ceteris planiusculis: suturâ distinctâ: spira depressiuscula, obtusa: peristomate simplici, acuto.

Axis 5 lin. Diam. 9½. Anfr. 6.

α. fasciâ continuâ.

β. fasciâ interruptâ.

Hab. in Maderæ sylvis; rarior.

9. Helix erubescens, *Prodr. MS.*

H. testa imperforata, globosa, tenui, rubescente: anfractibus strio-

lis rugisve valdè obliquis, sub-undulatis vel anastomosantibus corrugatis; primo vix sub-carinato; ceteris convexiusculis, æquis: spira elevato-obtusa: peristomate acuto, sub-reflexo, intus sub-incrassato, carneo.

Axis 4 lin. Diam. 7. Anfr. 5.

α. testâ fasciis maculisve fuscis ornatâ.—Tab. 5. f. 3.

β. testâ immaculatâ, unicolore.

Hab. in Maderæ sylvaticis.

10. Helix sub-plicata, *Sow.*—Tab. 5. f. 4.

Sow. in Zool. Journ. I. *p.* 56. *n°* 1. *t.* iii. *f.* 1 *!* (testa decorticata, semi-fossilis.)

Hab. in Insulâ quâdam parvâ, juxta Portum Sanctum, "Ilheo de Baxo" dictâ.

2. *Imperforatæ* (Depressæ).

Testa depressa umbilicata; umbilico omninò tecto. *Fer.*

11. Helix undata, *Prodr. MS.*—Tab. 5. f. 5.

H. testa juniore umbilicata, adulta imperforata, sub-globoso-depressa, unicolore, fusco-nigrescente: anfractibus corrugatis vel undato-rugosis, nitidiusculis; ultimo depresso, suprà planiusculo; ceteris convexiusculis, sutura distincta: spira brevi, obtusa, sub-depressa: peristomate simpliciusculo, sub-incrassato, vix reflexo, pallido.

Axis ½ poll. Diam. 1. Anfr. 6.

Helix corrugata, *Sol. MSS.; nec Gmel. nec. Dillw.*

Helix scabra, *Wood's Suppl. t.* viii. *f.* 62 *! nec Chemn., nec Lam. Feruss. &c.*

Hab. in Maderæ sylvis, graminosis montanis, &c., vulgaris.

12. Helix phlebophora, *Prodr. MS.*—Tab. 5. f. 6.

H. testa juniore umbilicata, hispida; adulta imperforata, sub-globosa, fusco bifasciata: anfractibus sub-tumidis, striis crebris, æqualibus, transversis, obliquis, sub-flexuosis sculptis; ultimo ad angulum peristomatis inferiorem depressiusculo, ventricoso, prominente: spira conoidea,

sub-exserta, obtusa ; sutura distincta, ab angulo peristomatis primò valde obliqua: apertura rotunda; peristomate continuo, simpliciusculo, paullùm incrassato: columella expansa, plana, rosea.

Axis 4—4½ lin. Diam. 8. Anfr. 5½.

Helix nivosa, *Sow. in Zool. Journ.* I. *p.* 56. *n°.* 3. *t.* iii. *f.* 3 !

Helix exalbida, *Wood, Suppl. t.* viii. *f.* 81 !

Hab. in Insulâ Portûs S^{a}; ubique vulgatissima.

Nomen alterum imponendum est ob priora (à testis quibusdam decorticatis, ut videtur, orta) speciei prorsus abhorrentia, ideoque difficultatem indagatoribus vel diligentissimis haud levem parantia. Nomen itaque novum, quodammodo aptius, è duobus incommodis minus esse malum videtur: tales enim mutationes pessimæ auctoritatis, nec nisi gravissimis argumentis probari possunt. In dilemmate verò tali, quis inter nominis veteris planè falsi et erronei adoptionem, et aptioris quamvis recentioris usum hæreret?

II. Subgenus, Helicodon. (Helicodonta *Fer.*)

I. *Personatæ.*

Peristoma sinuatum et incrassatum, vel reflexum atque dentatum, dentibus, laminis, plicisve tortuosis anfractûs penultimi partis convexæ sæpe coarctatum.

13. Helix arcta, *Prodr. MS.*—Tab. 5. f. 7.

H. testa rotundata, depressa, utrinque planiuscula, carinata, umbilico minimo perforata, solida, crassa, glabra: anfractibus striis crebris, æqualibus, transverso-obliquis crassiusculis rudibusve sculptis: spira convexo-depressa; sutura distincta, sub-impressa: apertura transversa, ovali, dente lamellata intùs ad ventrem* coarctata; peristomate albo, reflexo, continuo, æquali.

Axis 1—1¼ lin. Diam. 2—2¼. Anfr. 4—4½.

Hab. in Maderæ collibus aridis maritimis.

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* *Venter*, pars convexa anfractûs penultimi, aperturam (in Helicibus) coarctans.

14. Helix fausta, *Prodr. MS.*—Tab. 5. f. 8.

H. testa rotundata, carinata, sub-globoso-depressa, suprà* convexiore, pilis brevissimis undique scobinato-hispida, læviuscula: spira elevatiuscula, depresso-conoidea: anfractibus planiusculis, obsoletissimè transversè striatis; sutura distincta, impressa: apertura transversa, intùs angustata, exteriùs dilatata, dente lamellata intùs ad ventrem coarctata: peristomate extrà expanso, sub-reflexo, acuto; columellam versus albo, incrassato, sub-sinuato sc. obsoletè bidentato, reflexo, umbilicum penitùs obtegente.

Axis $1\frac{3}{4}$ lin. Diam. 3. Anfr. $5\frac{1}{2}$.

Hab. in sylvis Convallis "Boa Ventura" (i. e. Boni Successûs) dictæ, in Maderæ orâ Septentrionali.

Helici personatæ cl: *Draparnaudi* maximè quidem affinis, sed distinctissima.

15. Helix arridens, *Prodr. MS.*—Tab. 5. f. 9.

H. testa carinata, umbilico parvo perforata, rotundata, depressa, utrinque sub-planulata, tenui, hispida, læviuscula: spira convexo-depressa; anfractibus planiusculis, obsoletissimè transversè striatis; sutura sub-distincta: umbilico spirali†, rotundo: apertura edentula, transversa, intùs angustata et in umbilicum quasi cum rictu paullùm producta; peristomate interrupto, extrà simpliciusculo, sub-reflexo; angulum versus internum incrassato, sub-sinuato, albo, reflexo, et umbilicum partìm laminâ expansâ obtegente.

Axis $1\frac{1}{2}$ lin. Diam. 3. Anfr. $4\frac{1}{2}$—5.

Hab. in Madera.

Characteribus fortè artificiosis cum *Helicellis Hygromanibus* consocianda; sed affinitas summa cum priore reliquisque *Helicodontibus*

* *Suprà* latus quo umbilicus situs est; *infrà* quo spira, respicit.

† Umbilicus *spiralis* dicitur ubi plus minus anfractûs penultimi, antepenultimi, pluriumve intùs conspiciuntur. Huic opponitur umbilicus *cylindricus*.

naturalis, his adnumerare docet. *Helici edentulæ* cl: *Draparnaudi* proxima; caret autem impressionibus externis plicarum; caret quidem omninò plicis ipsis ullis: ideoque ab *Helice faustá* nostrâ et *H. personatâ Drap*ᵗ:, aliisque hujusce sectionis cognatis, nullo modo intervallo longo separanda est.

III. Subgenus, Helicigona.

1. *Carocollæ.* Umbilicus tectus.

16. Helix Webbiana, *Prodr. MS.*—Tab. 5. ff. 10.

H. testa adulta imperforata, tenui, nitida, sub-lampadiformi, depressa, carinata, utrinque convexa, corneo-fuscescente; suprà ad umbilicum virescente, convexiore, obliquè tenuiter striata, carinam versus utrinque sub-impressam, obtusam, suturamve granulis minutis scabra: spira convexo-depressa, obtusissima; sutura distincta; anfractibus planulatis, ultimo maximo: apertura transversa, sub-ovali, amplissima, patula, extrà carina angulata; peristomate interrupto, tenui, acuto; extrà valde expanso, patulo; ad columellam sub-incrassato, sub-reflexo.

Axis 3 lin. Diam. 9. Anfr. 3—3½.

Hab. in montibus Insulæ Portûs Sᵗⁱ.

Amico P. B. Webb, Armᵒ., Naturæ indagatori impigro ac peritissimo, speciem pulcherrimam atque rarissimam dico.

2. *Vortices.* Umbilicus apertus.

17. Helix Bulveriana, *Prodr. MS.*—Tab. 5. ff. 11.

H. testa rotundato-depressa, hemisphærica, rotata, suprà planulata, acutissimè carinata, tenui, nitidiuscula, tota minutissimè et confertim granulata, fusco-castanea, suprà fasciata: spira convexo-depressa, plus minus elevata, obtusissima; sutura obsoleta; anfractibus planis, æquis, quasi attritis vel confluentibus, ultimi carinâ acutissimâ, tenui, suprâ sulco exarata, limbata; umbilico patulo, spirali, profundo: apertura rotundato-lunata: peristomate interrupto, ad umbilicum incrassato, reflexo.

Axis 3—2½ lin. Diam. 7—8. Anfr. 8—7.

Helix Bulverii, *Wood, Suppl. t.* viii. *f.* 82*!*

Hab. in montibus Insulæ Portûs S^{ti}.

18. Helix tectiformis, *Sow.*—Tab. 5. f. 12.

Sow. in Zool. Journ. p. 57. *n°.* 6. *t.* iii. *f.* 6*!*

Hab. in insulâ quâdam parvulâ "Ilheo de Baxo" dictâ juxta Insulam Portûs S^{ti}.

19. Helix subtilis, *Prodr. MS.*—Tab. 5. f. 13.

H. testa orbiculari, utrinque depresso-planulata, tenui, unicolore, pallidè fusco, acutè carinata: spira depressa, sub-planulata; sutura distincta; anfractibus planulatis, striis transversis, obliquis, tenuibus, plus minus distinctis, æquidistantibus, interstitiisque striolis subtilissimis, crebris, decussatis: umbilico patulo, magno: apertura transversa, depressa, obliquè lunata; peristomate interrupto, sub-simplici, sub-reflexo.

Axis 1 lin. Diam. 3—4. Anfr. 5.

An Helix lenticula, *Feruss. Tabl. Syst. n°.* 154*?*

Hab. in Maderæ maritimis.

20. Helix actinophora, *Prodr. MS.*—Tab. 5. f. 14.

H. testa orbiculata, depressa, suprà convexiore sub-turgida, tenui, unicolore, fusco-rufescente, acutè carinata: spira convexiusculo-depressa, sub-planulata; sutura distincta; anfractibus planatis, striis creberrimis, tenuissimis, transversis undulatim laminosis, quibusdam ad carinam suturamve in laminas breves, membranaceas, lacinulasve acutas, radiantes productas, notatis: umbilico spirali, parvo: apertura transversa, rotundato-ovali, sub-lunata; peristomate interrupto, acuto, patulo, reflexo.

Axis 2 lin. Diam. 4. Anfr. 5.

Hab. in Maderæ sylvaticis.

IV. Subgenus, HELICELLA.

Lomastomæ; peristoma reflexum.

21. Helix pulchella, *Mull.*

Hab. in Madera.

22. Helix Porto-sanctana, *Sow:*

α. vulgaris.—Tab. 5. f. 15.

Sow: in Zool. Journ. I. *p.* 57. *n°.* 5. *t.* iii. *ff.* 5!

Hab. copiosissimè in Portu S^to^,

β? gigantea.—Tab. 5. f. 16.

Hab. in Portu S^to^.

An var. *β.* species potiùs? quamvis enim ab *α.* vix nisi magnitudine duplò ferè majore differt, status intermedios nunquam vidi Var. *α.* viva ubique copiosissima; *β.* rarissima nondum nisi statu semifossili, decorticato occurrit.

Aplostomæ; peristoma simplex.

* Verticilli.

23. Helix pusilla, *Prodr. MS.*—Tab. 5. f. 17.

H. testa rotundato-depressa, ecarinata, tenui, rufescente: spira convexiuscula; sutura distincta, impressa; anfractibus rotundatis, striis transversis, annularibus, elevatis, sub-membranaceis, tenuibus, remotis, æqualibus, plicatis; interstitiis striolis aliis exilissimis tenuissimisque creberrimis, spiralibus sc. transversas decussantibus, sculptis: umbilico patulo, spirali, profundo: apertura rotunda, vix lunata sc. circuli segmento perparvo dempto; peristomate simplici (tenui, acuto).

Axis $\frac{1}{2}$ lin. Diam. 1. Anfr. 4.

Hab. in Maderæ sylvis.

Obs. *Helici pygmææ* cl: *Drap:* quoad staturam et habitum maximè affinis; species autem revera distinctissima.

* * Hyalinæ.

24. Helix bifrons, *Prodr. MS.*—Tab. 5. f. 18.

H. testa rotundato-depressa, umbilicata, sub-carinata, tenui, nitida, concolore, corneo-virescente; suprà læviuscula, obsoletè striata; infernè striis valde distinctis sculpta: spira convexo-depressa; sutura distincta, impressa; anfractu ultimo infra carinam, ceterisque anfractibus, striis costisve transversis, æqualibus, crebris sulcatis: umbilico parvo, cylin-

drico, sub-spirali, profundo: apertura lunata; peristomate simplici, tenui, acuto, intùs albo marginato.

Axis 2½—3 lin. Diam. 6—7. Anfr. 7—8.

Hab. in Maderæ sylvis.

25. Helix cellaria, *Mull.*

Helix lucida, *Mont.*

Hellx nitens, *Drap.*

Hab. in Madera.

26. Helix crystallina, *Mull.*

Hab. in. Madera.

Heliomanes; peristoma marginatum ("bordé").

* Testa depressa vel globulosa.

27. Helix paupercula, *Prodr. MS.*—Tab. 5. f. 19.

H. testa rotundata, planata, suprà convexiore, umbilicata, sub-carinata, solidiuscula, rudi, ferè unicolore, anfractu ultimo supra carinam obsoletè fusco unifasciata: spira planata; sutura impressa; anfractibus rugosis, eroso-scrobiculatis, minutissime elegantissimèque granulatis; ultimo ad aperturam constricto: umbilico largo, patulo, spirali, profundo: apertura rotundata, coarctata; peristomate continuo, elevato-disjuncto, annulari, sub-patulo, acuto; labro intùs 1-dentato.

Axis 1 lin. Diam. 2—2½. Anfr. 3½—4.

Hab. in Maderæ et Portûs S^{ti}. maritimis.

28. Helix obtecta, *Prodr. MS.*—Tab. 5. ff. 20, a, b.

H. testa rotundata, depressa, infernè planata, suprà convexa, umbilicata, carinata, solidiuscula, rudi, albida, limo vel terra obducta: spiræ planulatæ anfractibus primis concavis, ceteris prominentibus turgidis; sutura distincta, valde impressa; anfractu ultimo ventricoso, carina distincta, utrinque sub-exarata vel sulco obsoletissimo expressa; omnibus rugosis, eroso-scrobiculatis, minutissimè elegantissimèque gra-

nulatis: umbilico mediocri, sub-spirali: apertura rotundata; peristomate continuo, sub-disjuncto, tenui, acuto, sub-expanso, intùs incrassato.

Axis 2 lin. Diam. 5. Anfr. 4½—5.

Hab. in montibus collibusque aridis Portûs S^{a}. rarior; copiosior in Insula "Ilheo de Baxo" dicta.

Præcedentis forsan status, vel varietas tantùm major.

29. Helix dealbata, *Prodr. MS.*

H. testa rotundato-depressa, utrinque convexa, umbilicata, carinata, solidiuscula, albida: spira convexo-depressa, sub-conoidea; sutura sub-distincta; anfractibus sub-planulatis, transversè rugoso-striatis, plerumque minutè granulatis; ultimo obtusè carinato: umbilico parvo, patulo, sub-spirali, minimè profundo; apertura rotunda, ochracea; peristomate continuo, reflexo.

Axis 2 lin. Diam. 4—4½, altera transversa 3—3½. Anfr. 6.

α. granulata; testa granulata.—Tab. 5. f. 21.

Hab. in montibus Portûs S^{a}.

β. lævis; testa egranulata, lævi, nitida.

Hab. in insulâ "Ilheo de Baxo" dictâ. Status *α*, solo calcareo ortus.

30. Helix maderensis, *Wood.*—Tab. 5. f. 22.

H. testa rotundato-depressa, utrinque planulata, umbilicata, carinata, solidiuscula; suprà læviuscula, fusco 1-fasciata; infernè striata: spira convexiuscula, sub-planulata; sutura distincta; anfractibus planatis, infrà transversè striatis, ultimi ad aperturam granulati carina acuta, striis suprà carinam obsoletis: umbilico lato, patulo, spirali: apertura rotunda; peristomate continuo, circinato, annulari sub-disjuncto, crassiusculo, sub-expanso.

Axis 1½ lin., rariss. 2; Diam. 3, rariss. 4. Anfr. 6—7.

Helix maderensis, *Wood. Suppl. t.* viii. *f.* 84!

Hab. in Maderâ; vulgatissima.

31. Helix compar, *Prodr. MS.*—Tab. 5. f. 23.

H. testa rotundato-depressa, utrinque planulata, umbilicata, sub-carinata, solidiuscula, fusco bifasciata, utrinque plicato-costata: spira

convexiuscula, sub-planulata; sutura distincta, impressa; anfractibus convexiusculis, plicis vel striis transversis, elevatis, acutis, distinctis, crebris, æquidistantibus æqualibusque costatis; interstitiis lævibus; ultimi carina obtusa: umbilico lato, patulo, spirali, profundo: apertura rotundato-ovali; peristomate continuo, circinato, sub-disjuncto, crassiusculo, reflexo.

Axis 1½ lin. Diam. 3¼. Anfr. 6.

Hab. in Maderæ collibus maritimis; rariss.

32. Helix leptosticta, *Prodr. MS.*—Tab. 5. f. 24.

H. testa rotundato-depressa, umbilicata, sub-carinata, nitidiuscula, tenui, pallidè cornea, obsoletè fasciata: spira convexo-depressa; sutura distincta; anfractibus convexis, sub-striatis, minutè et elegantissimè reticulato-granulatis; ultimi carina obtusa: umbilico patulo, spirali: apertura rotundato-ovali; peristomate continuo, simpliciusculo, sub-incrassato, sub-reflexo.

Axis 1½ lin. Diam. 3. Anfr. 5—5½.

Hab. in Maderæ collibus maritimis.

33. Helix lentiginosa, *Prodr. MS.*—Tab. 5. f. 25.

H. testa rotundato-depressa, suprà sub-planulata, umbilicata, sub-carinata, tenui, maculata et sub-fasciata: spira convexo-depressa; sutura distincta; anfractibus convexiusculis, striato-scobinatis vel squamuloso-cancellatis, striis sc. interruptis squamiformibus, lunatis, quincuncialibus sculptis: umbilico mediocri, sub-patulo, spirali: apertura transversè ovali, sub-lunata; peristomate interrupto, reflexo.

Axis 1½ lin. Diam. 2½—3. Anfr. 5.

Hab. in Maderæ rupibus maritimis.

Helici arridenti nob: affinis.

34. Helix calva, *Prodr. MS.*—Tab. 5. f. 26.

H. testa rotundato-globulosa, sub-depressa, imperforata, vix sub-carinata, nitidiuscula, sub-tenui, sub-pellucida, obsoletissimè 2-fasciata: spira convexa, elevatiuscula; sutura distincta; anfractibus planiusculis,

transversè costulato-striatis, striolisque spiralibus obsoletissimis, subtilissimis exilissimisque, æquis notatis; ultimo ferè ecarinato, ad aperturam ochraceo, suprà nitido, lævi: umbilico clauso: apertura transversa, multò latiore quàm alta, sub-lunata, intùs angustata, extrorsùm ampliore; peristomate longè interrupto, incrassato, sub-reflexo.

Axis 2—2¼ lin. Diam. 3½—4. Anfr. 6½—7.

Hab. in Maderæ sylvis.

Helici edentulæ Drap^{di}. aliquatenùs formâ affinis; sed vix HELICODONTIBUS releganda.

HELICELLIS *Hyalinis* affinitate naturali, et præ ceteris *H. bifronti* nostræ accedit; peristomate verò marginato, striisque subtilissimis tenuissimisque spiralibus prorsùs aliena.

35. Helix abjecta, *Prodr. MS.*—Tab. 6. f. 1.

H. testa parvula, rotundato-pyramidata, conoidea, carinata, umbilicata, crassa, solida, rudiuscula, utrinque scabra vel granulosa, rugosa, supra carinam fusco pallidè unifasciata: spira convexa, conoidea; sutura distincta; anfractibus compactis, convexiusculis, transversè rugosis et granulatis: ultimi carina sub-acuta, ad suturam approximata: umbilico parvo, spirali, profundo: apertura rotundata; peristomate continuo, reflexo.

Axis 1¾—2 lin. Diam. 3—3½. Anfr. 6½—7.

Hab. in insulâ Portûs S^{ti}., unà cum *H. compacta* degens; vulgatissima.

Inter *H. compactam* et *H. echinulatam* nostram quasi intermedia; ab utrâque satis dìstincta: priorì verò quàm maximè affinis.

36. Helix compacta, *Prodr. MS.*—Tab. 6. f. 2.

H. testa parvula, rotundato-globulosa, sub-conoidea, perforata, subcarinata, crassa, solida, rudiuscula; infrà scabra, rugosa; suprà lævìore, nitidiuscula, pallidiore, fusco obsoletè 1-fasciata: spira convexa, elevatiuscula; sutura distincta; anfractibus compactis, planiusculis, transverse striatis et granulatis; ultimo sub-carinato, supra angulum læviusculo

sc. egranulato, umbilicum coarctante: umbilico minimo, sub-spirali, rimæformi: apertura rotundato-lunata; peristomate interrupto, (labris approximatis, aliquando sub-continuis) sub-reflexo.

Axis 2 lin. Diam. 3. Anfr. 6—6½.

Sowerb. in Zool. Journ. I. *t.* iii. *f.* 8 *!*

Hab. in Insula Portûs S^{ti}. gregaria, ubique copiosissima: in Maderâ ad Promontorium S^{ti}. Laurentii ("Ponta Saô Lourenço") solùm.

37. Helix consors, *Prodr. MS.*—Tab. 6. f. 3.

H. testa rotundato-depressa, perforata, vix sub-carinata, crassiuscula, solida, rudiuscula; infrà præsertim scabra, rugosa; suprà lævìore; utrinque pallido fuscoque variata, ad aperturum ochracea: spira convexo-depressa; sutura sub-indistincta; anfractibus planatis, transversè striatis et granulatis; ultimo sub-carinato, umbilicum coarctante, granulis supra angulum obsoletis: umbilico minimo, sub-spirali, rimæformi: apertura rotundato-lunata; peristomate distinctè interrupto, sub-reflexo.

Axis 2½—3 lin. Diam. 4½—5. Anfr. 6—6½.

Hab. in Insulâ Portûs S^{ti}. cum præcedente; rarior.

Præcedenti vel maximè affinis; characteres itaque extricatu difficillimæ: sed forma magis depressa numerusque anfractuum isdem, quamvis testa ferè duplò major, speciem esse distinctam suadent; obstante nullâ differentiâ loci, soli, cibi, nec aliâ quâpiam hujusmodi causâ quæ talem mutationem efficere posset.

38. Helix depauperata, *Prodr. MS.*—Tab. 6. f. 4.

H. testa rotundato-depressa, umbilicata, ecarinata, tenuiuscula, suprà convexa, unicolore, sordida, minutissimè et elegantissimè confertìm reticulato-granulata: spira convexo-depressa; sutura distincta, subimpressa; anfractibus convexis, sub-tumidulis, transversè sub-striatis; ultimo sub-rotundato: umbilico mediocri, aperto, spirali, profundo:

apertura rotundata; peristomate sub-continuo, simpliciusculo, tenui, intùs sub-marginato.

Axis 2—2½ lin. Diam. 4—4½. Anfr. 5—5½.

Hab. in montibus Insulæ Portûs S^{a}.

39. Helix lurida, *Prodr. MS.*—Tab. 6. f. 5.

H. testa sub-globosa, depressiuscula, suprà convexa, umbilicata, ecarinata, tenuiuscula, fusco sub-fasciata, nitidiuscula: spira convexo-depressa; sutura distincta; anfractibus convexis, minutissimè et obsoletissimè confertìm reticulato-granulatis; ultimo rotundato, juxta suturam granulato, supernè lævi sc. egranulato: umbilico parvo, cylindrico, profundo, aperto: apertura lunata, sub-ovali; peristomate simplici.

Axis 3 lin. Diam. 5. Anfr. 5½—6.

Hab. in montibus Insulæ Portûs S^{a}.

Sequenti proxima.

40. Helix nitidiuscula, *Sow.*—Tab. 6. f. 6.

Sow. in Zool. Journ. I. *p.* 57. *n°.* 4. *t.* iii. *f.* 7 *!*

Hab. in Maderâ et Portu S^{to}; ubique vulgatissima.

41. Helix punctulata, *Prodr. MS.*—Tab. 6. ff. 7, 8.

α. setulosa; testa sub-tenui, sub-inflata, scabra, spinelloso-hispida.

Axis ½ poll. Diam. ⅜. Anfr. 5.—Tab. 6. f. 7.

Helix punctulata, *Sow. in Zool. Journ.* I. *p.* 56. *n°.* 2. *t.* iii. *f.* 2 *!*

β. solida; testa solida, glabriuscula, pallida.

Axis ⅜ poll. Diam. ½. Anfr. 5.—Tab. 6. f. 8.

Hab. in Portu S^{to}.

An satis ab. *H. nitidiusculâ Sow.* distincta?

42. Helix pisana, *Mull.*

H. rhodostoma *Drap.*, cingenda *Mont.*, &c.

Hab. in Maderæ Promontorio "P.ta. Saô Lourenço" dicto. In Portu S.to. vinetarum calamitas.

43. Helix lauta, *Prodr. MS.*—Tab. 6. f. 9.

H. testa sub-globosa, suprà convexa, umbilicata, ecarinata, tenuiuscula, (alba, fasciis angustis, interruptis, fuscis, obsoletis ornata), nitidiuscula: spira convexo-depressa, sub-elevata; sutura distincta; anfractibus convexis, striis confertissimis, æqualibus, concinnis, transversis sculptis; ultimo rotundato: umbilico parvo, cylindrico, profundo, aperto: apertura lunata, sub-rotunda; peristomate acuto, intùs annulo distincto, elevato, margini approximato.

Axis $\frac{1}{4}$ poll. Diam. $\frac{7}{16}$. Anfr. 5.

Hab. in Portu S.to.

Specimen unicum decorticatum tantùm habeo, à Rev.o. Dom.o. Bulwer repertum, quod mihi cl. G. B. Sowerby humanissimè communicavit. *H. luridæ* nostræ, necnon *H. striatæ Drap.:* (*caperatæ Mont.:*) et forsan aliis quibusdam proxima: sed ab omnibus distincta videtur.

44. Helix striata, *Drap.d.?*

Hab. in Madera; rariss.

Differt umbilico et numero anfractuum, pro magnitudine, majore. Quum autem testas perpaucas easque nondum adultas adhuc repertas habeo, distinguere vix audeo.

45. Helix, rotula, *Prodr. MS.*—Tab. 6. f. 10.

H. testa rotundata, conoideo-depressa, suprà sub-planulata, subperforata, carinata, scabra, nitidiuscula, fasciata: spira conoidea, obtusissima; sutura obsoleta; anfractibus planis, transversè striatis et granulatis; ultimo acutè carinato, carina ad peristoma obsoleta: apertura lunata, extrorsum dilatata; peristomate intùs incrassato, acuto, sub-expanso; ad angulum internum reflexo, calloso, perforationem obtegente.

Axis 3 lin. Diam. 6. Anfr. 8.

Hab. in montibus Portûs S.ti.

46. Helix polymorpha, *Prodr. MS.*—Tab. 6. ff. 11—16.

H. testa rotundato-depressiuscula, umbilicata, carinata, crassiuscula, solida, fusco fasciata et maculata: spira conoideo-depressa, aliquando ferè planata, granulata; anfractibus planiusculis; primorum saltem sutura obsoleta; ultimi carina plùs minùs acuta: umbilico patulo, spirali, largiusculo: apertura lunato-rotundata; peristomate sub-reflexo.

α. *irrasa;* testa depresso-conoidea, sub-globulosa, utrinque granulato-scaberrima, limo vel terra obducta: spira convexo-elevatiuscula, conoidea; anfractibus convexis; sutura distincta; carina obtusa: peristomate sub-interrupto.

Axis 3 lin. Diam. 5. Anfr. 8.

Albida, fasciis fuscis distinctis, superiore lato, continuo, distinctissimo; infrà sc. spira sub-maculata, variegata.

f. 11, l. c.

Hab. in solo rubro "Tufa" Geologicis dicto, ad promontoriam S^{ti}. Laurentii Maderæ.

β. *depressiuscula;* testa rotundato-depressiuscula, obsoletè utrinque granulata, suprà præsertim nitidiuscula, læviuscula sc. granulis raris, obsoletis: spira convexo-depressiuscula; anfractibus convexiusculis; sutura distincta; carina obtusa: peristomate interrupto.

Axis 2½ lin. Diam. 5—5½. Anfr. 7.

Supra albida, fasciis fuscis, superiore lato, continuo, distincto, ceteris interruptis vel obsoletis; infrà sc. spira albido fuscoque maculata, variegata.

f. 12, l. c.

Hab. in solo *Tufa* dicto in collibus maritimis prope urbem Funchalensem Maderæ.

γ. *arenicola;* testa rotundata, supra sub-planulata, utrinque granulata; suprà præsertim nitida, granulis obsoletioribus: spira convexo-

depressiuscula, plùs minùs elevata; anfractibus convexiusculis; sutura distincta; carina sub-obtusa: peristomate sub-continuo.

Axis 2—2½ lin. Diam. 4½—5. Anfr. 7.

Sub-var. 1. Suprà fusco fasciata; fascia superiore distincta, sub-continua, angusta; ceteris interruptis.

f. 13, l. c.

2. Inornata sc. non fasciata, variegata.

Albida, nitida, spira fusco maculata, variegata. Colores quodammodo lætiores quàm in ceteris; albo præsertim clariore.

Status *a*, è solo calcareo ortus.

Hab. in arenosis calcareis Promontorii S^ti. Laurentii Maderæ.

δ. attrita; testa rotundato-depressa, rotata; infrà planulata; suprà convexa, nitidiuscula; utrinque confertìm granulata: spira convexo-planata; anfractibus planis, quasi attritis; sutura obsoletissima; carina acutissima: umbilici margine (præsertim in junioribus) abrupto, declivi: peristomate ferè interrupto.

Axis 2 lin. Diam. 4½—5½. Anfr. 7.

Sub-var. 1. Suprà pallida, fusco fasciata; fascia superiore angusta; plerumque unica.

f. 14, l. c.

Helix tectiformis, *Wood. Suppl. t.* viii. *f.* 83 *!*

2. Tota fusca, sub-unicolor, præter spatium vel fasciam latam pallidam circa umbilicum.

3. Tota variegata, nec fasciata.

Sub-varietas quæque colore magis fusco quàm in ceteris gaudet: in 1ma sordidè albido vel pallidè ochraceo fuscoque variegata et maculata; anfractûs ultimi pars semper in omnibus juxta peristoma ochracea, immaculata.

Hab. in collibus montibusve Portûs S^ti.

ε. *calcigena;* testa rotundato-depressa; suprà planulata, lævi, nitida, ad aperturam tantùm sub-granulata: spira convexo-depressa, plûs minùs elevata, granulata; anfractibus planatis; ultimi sutura impressa, ceterorum obsoleta; carina sub-acuta: peristomate subinterrupto.

Axis 2—2½ lin. Diam. 5—5¾. Anfr. 7½.

Sub-var. 1. Suprà tota alba; spira albida fusco variegata.

f. 15, l. c.

2. Suprà fasciata; spira albida fusco variegata.

Status δ. vel ζ., solo calcareo ortus.

Hab. in solo calcareo Insulæ cujusdam, "Baxo" dictæ, juxta Portum S^{tum}.

ζ. *pulvinata;* testa rotundato-conoidea, utrinque confertim granulata: spira elevata, conica, anfractui ultimo quasi superimposita; anfractibus (præter primos) convexiusculis, ultimo tumidulo; sutura distincta, impressa; carina sub-obtusa: peristomate continuo.

Axis 2½—3 lin. Diam. 5. Anfr. 7½.

Sub-var. 1. Suprà tota alba spira sub-maculata.

2. Suprà fusco fasciata; spira maculata.—f. 16, l. c.

Colores in utroque statu (sc. sub-varietate) quàm in ceteris varietatibus longè pallidiores. Testa quidem in omnibus pallida, albida, apice spiræ fusco.

Hab. in montibus collibusve Portûs S^{u}.; cum δ. *attrita* nostrâ degens.

Varietates δ. et ζ (forsan etiam ε) primo aspectu distinctissimæ, tot forsan species constituendæ quibusdam videantur. Adeò tamen, mediante ε., sunt conjunctæ, ut tres illæ δ, ε, ζ nec à seipsis nec ab α, β, γ, quibus ordine inverso analogæ sunt, separari debent. Sed in re tam dubiâ, non is sum qui cuilibet meas varietates pro speciebus habenti, increparem.

* * Testa trochoidea, carinata.

47. Helix cheiranthicola, *Prodr. MS.*—Tab. 6. f. 17.

H. testa pyramidata, conoidea, umbilicata, carinata, solidiuscula, tota scabra, plerumque fasciata: spira elevata, pyramidata, obtusa; sutura distinctissima, impressa; anfractibus convexis, tumidis, distinctis, confertìm granulatis; ultimi carina obtusa: umbilico mediocri, patulo, spirali profundo: apertura rotundata; peristomate continuo, sub-disjuncto sc. circinato, incrassato, sub-reflexo.

Axis 3 lin. Diam. 4. Anfr. 8.

Sub-var. 1. *zonata;* suprà fasciata: spira fascia, unica, lata, juxta suturam: carina albida.

f. 17, l. c.

2. *maculata;* suprà fasciata: spira maculata vel variegata, nec fasciata.

3. *albida;* tota albida, nec fasciata: spira sub-maculata.

Hab. in arbusculis *Cheranthi tenuifolii Herit:* in monte Portûs S^{a}. quodam "Pico branco" dicto: et in Insulâ "Ilheo de Baxo" dicto, sed rarissima.

Varietati ζ. *pulvinatæ Helicis polymorphæ* nimis forsan affinis: sed formâ et anfractibus tumidis et sutura impressa dignoscitur.

48. Helix oxytropis, *Prodr. MS.*—Tab. 6. f. 18.

H. testa depresso-conoidea, suprà planulata, perforata, carinata, tota scabra, fusca, sub-fasciata: spira depresso-conica; sutura distincta; anfractibus planiusculis; ultimi carina acuta, distinctissima, suprà marginata sc. exarata vel sulco expressa; omnibus distinctissimè confertìm granulatis, asperis: umbilico minimo, sub-spirali, aperto: apertura rotundata; peristomate continuo, circinato, disjuncto, reflexo.

Axis 2$\frac{1}{4}$ lin. Diam. 4. Anfr. 6$\frac{1}{2}$.

Hab. in collibus maritimis Portûs S^{a}.

49. Helix echinulata, *Prodr. MS.*—Tab. 6. f. 19.

H. testa parvula, conoidea, sub-pyramidata, depressiuscula, suprà planulata, perforata, carinata, tota scaberrima, fusca, suprà fasciata:

spira pyramidata elevata; sutura distincta, impressa; anfractibus convexis; ultimi carina acuta, distincta, suprà marginata sc. sulco expressa vel exarata; omnibus granulis distinctissimis, confertis, asperrimis scobinatis et quasi echinulatis: umbilico parvo, sub-spirali, aperto; apertura rotundata; peristomate continuo, circinato, disjuncto, reflexo.

Axis 2 lin. Diam. 2½. Anfr. 6.

Hab. in monte "Pico branco" dicto Insulæ Portûs S^{ti}.

Species elegantissima.

50. Helix duplicata, *Prodr. MS.*—Tab. 6. f. 20.

Helix bicarinata, *Sow. in Zool. Journ.* I. *p.* 58. *n°.* 7. *t.* iii. *f.* 7! *Wood, Suppl. t.* viii. *f.* 85! *non Feruss.*

Monstrosa; anfractu ultimo disjuncto; sutura profunda, excavata.

Hab. in Insulâ Portûs S^{ti}.

Nomen ægrè, et quasi coactus, mutavi; ob *Helicem C. bicarinatam* cl: *Ferrussaci, Tabl. Syst. n°.* 350.

51. Helix turricula, *Prodr. MS.*—Tab. 6. f. 21.

H. testa turrita, pyramidata, sub-cylindrica, bicarinata, perforata, tota minutè et confertissimè granulata, fusca, ferè unicolore, vel suprà obsoletè fasciata: spira valde elevata, obtusissima; sutura distincta; anfractibus bicarinatis, carinis æqualibus, prominentibus, distinctis, sulco divisis: apertura rotunda; peristomate continuo, circinato, disjuncto, tenui, reflexo.

Axis 4 lin. Diam. 3. Anfr. 8—8½.

Hab. in Insulâ quâdam "Ilheo de Cima" dictâ, juxta Insulam Portum S^{tum}.

Species notabilior, elegans.

52. Helix bicolor, *Prodr. MS.*—Tab. 6. f. 22.

H. testa globuloso-conoidea, sub-imperforata, lævius cula, nitida, vix sub-carinata, fasciis albis fuscisque lætè-coloribus ornata: spira elevatiuscula, obtusissima; sutura distincta; anfractibus sub-planulatis, trans-

versè striatis: apertura extrorsùm ampliore; peristomate longè interrupto, tenui, simpliciusculo, intùs ad angulum incrassato, reflexo, perforationem minimam ferè obtegente.

Axis 2 lin. Diam. 3. Anfr. 7.

Hab. in summo cacumine montis "Pico de Facho" dicto Portûs S^{u}.

Species nitidissima, coloribus distinctissimis sc. fasciis lætè coloratis gaudens. Ob affinitatem *Helici maritimæ Drap.*, cel: Ferussaco obseeutus, huc relegavi; sed ambæ potiùs priori sectioni post *Helicem variabilem Drap.* (*H. virgatam, Mont.*) inserendæ sunt.

† Evolutatæ, *Cochloides.*

* Apertura ferè edentula.

1. *Columella solida.*

———————, planata et ad basin truncata.

5. Sub-genus, *Cochlicopa.*

Styloides; *testa turrita, apertura brevi, &c.*

53. Helix C. acicula, *Fer.*

Buccinum Acicula, *Mull.*—Bucc. terrestre, *Mont.*—Bulimus Acicula, *Brug. et Drap.*—Achatina Acicula, *Lam^{k} et Nills.*

Hab. in Madera.

54. Helix *C.* tornatellina, *Prodr. MS.*—Tab. 6. f. 23.

H. testa obovato-oblonga vel obconico-cylindrica, lævi, nitida, corneo-rufescente (castanea): spira breviuscula, obtusa, duas partes ex quinque totius longitudinis æquante; anfractibus planis; sutura obsoleta: apertura longitudinali, coarctata, posticè valde angustata; labro anticè producto, porrecto, sub-inflexo, posticè sub-sinuato: columella prominula, abruptè et obliquè truncata, torta; plica in ventrem longitudinali, sub-obsoleta, callosa, labro adversa, ad partem posticam angustatam aperturæ, hanc coarctante.

Long. 4—5 lin. Diam. 2—2¼. Anfr. 7.

$\frac{\text{Spira}}{\text{Apertura}}$, $\frac{2}{5}$—$\frac{2}{3}$.

Hab. in Madera.

Hæc et 3 forsan sequentes *Helici folliculo* affines.

55. Helix C. melampoides, *Prodr. MS.*—Tab. 6. f. 24.

H. testa obovato-oblonga, ... : spira breviuscula, obtusissima, partes duas ferè ex quinque totius longitudinis æquante; anfractibus planis, ultimo sub-ventricoso; sutura obsoleta: apertura longitudinali, anticè effusa, sub-patula, omninò edentula; labro recto, æquali: columella obsoleta, obliquè truncata.

Long. 5¼ lin. Diam. 2¼. Anfr. 6.

$\frac{\text{Spira}}{\text{Apertura}}$, $\frac{4}{7}$.

Hab. in Insulâ quâdam, Portum Sanctum ab oriente spectante, "Ilheo de Cima" dictâ. *v. m.*

Priori nimìs affinis, et forsan varietas tantùm; at major, aliquantulum ferè ventricosior, apertura semper edentula, anticè magìs effusa, posticè minus angustata, columella obsoletiore, et labro recto, æquali, nec sinuato, nec anticè producto. Testa decorticata, crassa, solida, opaca; sed hæc etiam in priore (*H. tornatellina*), post mortem anïmalis obtinent: vivam nondum vidi.

56. Helix C. triticea, *Prodr. MS.*—Tab. 6. ff. 25, 26.

H. testa obovato-cylindrica, sub-gracili, sub-conica, nitida, lævi: spira acutiuscula, dimidium testæ æquante; anfractibus planis; sutura obsoletiuscula: apertura obovata, biplicata; plica altera transversa, inter columellam et angulum labri in medio posita, altera magìs interna minore in columellam; duabus aliquando obsoletis; columella anticè lata, sub-expansa, plana, vix truncata, in labrum simplex rectum æquale attenuata.

Long. 3 lin. Diam. 1½. Anfr. 6.

α. *biplicata;* apertura 2-plicata.—f. 25, l. c.

β. *edentula;* plicis obsoletis.—f. 26, l. c.

Hab. in Portu S^to^.

57. Helix C. ovuliformis, *Prodr. MS.*—Tab. 6. f. 27.

H. testa angusto-elliptica, sub-pupæformi, diametro utrinque æquali, abbreviata, nitida, lævi: spira obtusissima, dimidium testæ æquante; anfractibus convexiusculis, sub-tumidis; sutura distincta: apertura obovata, angusta, biplicata; plica altera transversa, abrupta, prominente, inter columellam et angulum labri in medio posita; altera ad columellam, magis obsoleta, obliqua: columella expansa, tenui, torta, obliquè truncata.

Long. 2 lin. Diam. 1. Anfr. 4.

Hab. in cacumine montis "Pico de Facho" in Insula Portûs S^ti^.

58. Helix C. gracilis, *Prodr. MS.*—Tab. 6. f. 28.

H. testa elongato-obovata, gracili, tenui, vitrea, nitida, lævi, (imperforata): spira sub-attenuata, obtusa, dimidium testæ excedente; anfractibus planiusculis; sutura obsoletiuscula: apertura obovata, edentula: columella lata, expansa, vix truncata, in labrum tenue, sub-marginatum attenuata.

Long. 2 lin. Diam. 1. Anfr. 5.

Hab. in monte "Pico Branco" Insulæ Portûs S^ti^.

Facies *Helicis* (*Cochlicellæ*) *Clavuli Fer.* (*H. Goodalli, Mill. Annals of Philos.*); sed magis turrita; anfractu ultimo cum penultimo majore; sutura obsoletiuscula, non distincta, impressa; testa lævissima, imperforata, nec striata, nec sub-perforata. Inter *Helicem triticeam* et *Helicis lubricæ* varietatem nostram quodammodo media, ab utraque distincta.

59. Helix C. lubrica, *Mull.*—Tab. 6. f. 29.

Var. testa aperturaque angustiore, minùs ventricosa, magis elongata.

Hab. in Maderâ.

2. *Testa perforata vel umbilicata &c.; peristomate simplici.*

a. Anfractibus æqualibus, ultimo ceteris omnibus breviore.

6. Sub-genus, *Cochlicella*.

60. Helix C. ventrosa, *Fer.*
Bulimus ventricosus, *Drap.*
Hab. in Maderâ et Portu S^to^.

61. Helix C. decollata, *Linn.*
Bulimus decollatus, *Drap.*
Hab. in Maderâ.

* * Apertura ferè dentata vel laminata.

1. *Ecanaliculatæ; peristomate plerumque non continuo.*

7. Sub-genus, Cochlodon. (Cochlodonta *Fer.*)

1. Testa cylindrica.

62. Helix C. anconostoma, *Prodr. MS.*—Tab. 6. f. 30.

H. testa cylindrica, pupæformi, læviuscula, nitida, corneo-rufescente: spira obtusa; anfractibus convexis, rotundatis, æquis, striis transversis, obliquis, obsoletis, indistinctis; sutura distincta, impressa: apertura 1-dentata, elliptica, sub-angustata, longiore quàm lata, sub-trigona, anticè angulata: columella recta, suprà cubito vel flexurâ abrupto, acuto, cum labro tenui, reflexo conjuncta: dente lamellato in ventrem juxta labrum obsoletiusculo, à labro distincto.

a. gyrata; testa elongata: aperturæ cubito distinctissimo. f. 30. l. c.
Long. $1\frac{3}{4}$ lin. Diam. 1. Anfr. 7.

β. curta; testa abbreviata: aperturæ cubito obsoletiore.
Long. $1\frac{1}{4}$ lin. Diam. $\frac{3}{4}$. Anfr. 6.
Hab. in Madera.

Helici C. umbilicatæ, Fer. (*Pupa umbilicata, Drap., Lam^k.*; *Turbo muscorum, Mont.* t. xxii. f. 3.) proxima, præsertim per varietatem (rariorem) *β. curtam;* sed distincta videtur. In Icone cl: Montagui suprà indicatâ, testa Britannica ejusque characteres à nostrâ Maderensi optimè distinguuntur.

63. Helix C. cheilogona, *Prodr. MS.*—Tab. 6. f. 31.

H. testa sub-ovata, cornea, lævi, vel obsoletè striata: apertura 3–plicata, coarctata, anticè prominula; plica unica in columellam; duabus parallelis in ventrem positis; intermedio minore: labro expanso, intùs marginato, sinuato-angulato: umbilico magno, patulo, profundo.

Long. $1\frac{3}{4}$ lin. Diam. 1. Anfr. 6.

Hab. in Madera.

64. Helix C. sphinctostoma, *Prodr. MS.*—Tab. 6. f. 32.

H. testa cylindrica, fusca: anfractibus planis transversè sub-striatis: apertura 4–6–plicata; plicis duabus in columellam, postica obsoleta; duabus in ventrem, quarum anterior plicæ anteriori columellari æqualis, posterior magna, complicata, cum dente ad angulum inferiorem labri posito in unum conjuncta: labro reflexo, posticè sub-angulato vel sinuato, ad angulum intùs dentato, anticè 1–2–plicata: umbilico patulo, profundo.

Long. 2 lin. Diam. 1 Anfr. 7.

Hab. in Madera.

Testa plùs minùs striata.

65. Helix C. monticola, *Prodr. MS.*—Tab. 6. f. 33.

H. testa cylindrica, castanea, pallido fasciata: anfractibus convexis, tumidis, striis elevatis, æquidistantibus, transversis sculptis; sutura impressa: apertura sub-sexdentata; columella 2-plicata, plica posteriore obsoletissima; plicis duabus approximatis, parallelis, in ventrem, quarum anterior minor; posterior magna, cum labro continua: labro sub-reflexo, æquali, 3–plicata; plica intermedia majore; anteriore et posteriore minutis.

Long. $1\frac{1}{2}$ lin. Diam. vix 1. Anfr. 6.

Hab. in summo cacumine Montis "Pico de Facho" Insulæ Portùs S^{ti}.

Obs. Priori (*H. sphinctostomati*) quoad plicas affinis; sed distincta.

66. Helix C. calathiscus, *Prodr. MS.*—Tab. 6. f. 34.

H. testa cylindrica, ovoidea, abbreviata, castanea, pallido fasciata: anfractibus convexis, sub-tumidis, costulis æquidistantibus, transversis, crebris, sculptis; sutura impressa: apertura sub-septemplicata; columella 1-plicata; plicis duabus in ventrem, quarum anterior valde interna, minuta, dentiformis; altera posterior magna, cum labro continua: labro expanso, sub-sinuato; callo intùs margini parallelo, posticè in dentem duplicem desinente, anticè dente minuto, simplici, obsolete et plica unica intermedia, magna, instructo.

Long. $1\frac{3}{4}$ lin. Diam. 1. Anfr. 7.

Hab. in summo cacumine montis "Pico de Facho" Portûs S^{ti}.

67. Helix C. cassida, *Prodr. MS.*—Tab. 6. f. 35.

H. testa ovata, ventricosa, abbreviata, sub-imperforata: anfractibus planis, striis elevatis, crebris, æquidistantibus, transversis; sutura subindistincta: apertura 7–8-plicata; columella biplicata, plica posteriore minore; plicis duabus sub-æqualibus, parallelis in ventrem positis, exteriore paullò majore, cum labro continua, sinum efficiente: labro expanso, 5-plicata; plica anteriore minore, aliquando obsoleta; tribus intermediis lateralibus, approximatis, superiore magna, duabus inferioribus minoribus, quarum infima sub-dentiformis; quinta infima minima, ad angulum labri posita: perforatione minima.

Long. 2 lin. Diam. $1\frac{1}{2}$. Anfr. 7—8.

Hab. in Maderæ convallibus, in rupibus aridis umbrosis.

Recens semel tantum lecta; necdum vivam vidi. Ad locum "Caniçal" dictum, inter alias plurimas Helicis species* paullò frequentior, sed statu semifossili.

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* Testæ illæ, hìc et in Portu S^{to}., statu *semi-fossili*, in arenâ calcareâ, inter concreta ramiformia (minimè "Lignites"), repertæ, omnes terrestres, plurimæ (forsan omnes) etiam hodiè in Maderâ vel Portu S^{to}. vivunt; nec ullam quidem speciem marinam cum illis commixtam vidi. "*Delphinula sulcata Lam^{k}.?*" *Bowd. Exc. p.* 140. f. 33. a, b, est *Helicis* species, (*Helix Delphinula* nob.) valde elegans, *Helici tectiformi Sow.* affinis. A Delphinulâ prorsùs aliena.

VI. Genus, CLAUSILIA, *Drap*^{d}.

Helicis sub-genus *Cochlodina*, Fer.

68. Clausilia crispa, *Prodr. MS.*—Tab. 6. f. 36.

C. testa turrita, sub-ventricosa: anfractibus convexiusculis, striis transversis, creberrimis, minutissimè flexuosis sculptis, interstitiis elegantissimè decussatim punctulato-striatis, quasi cancellatis; sutura distincta, impressa: apertura oblonga, biplicata; plicis columellaribus, approximatis, divaricatis, sub-posticis, postica prominente sc. extrorsùm ad marginem peristomatis producta eique continua, sinum ad angulum posticum aperturæ formante: peristomate simplici, acuto, sub-expanso: costis dorsalibus rimaque umbilicali obsoletis.

Long. 7 lin. Diam. 2. Anfr. 9.

Hab. in rupibus sylvarum Maderæ.

Peristomate nec elevato neque disjuncto sc. columellari obsoleto, necnon costis duabus dorsalibus rimaque umbilicali sub-nullis, quin et quodammodo forma et magnitudine ad *Clausiliam bidentem Drap.* (*Turbinem laminatum Mont.*) magis quàm ad aliam quampiam speciem accedit.

69. Clausilia deltostoma, *Prodr. MS.*—Tab. 6. f. 37, 38.

C. testa turrita, gracili, obtusa: anfractibus planiusculis, striis rectiusculis, crebris, elevatis sculptis: apertura obliquè obovato-rotundata, deltoidea, effusa, posticè angustata, sub-biplicata; plica antica columellari, interna, obliqua, duplici; postica simplici, prominente sc. extrorsùm ad marginem peristomatis producta eique continua, sinum ad angulum posticum aperturæ efficiente: peristomate continuo, expanso, reflexo, disjuncto.

Long. 5—5½ lin. Diam. 1½. Anfr. 10—11.

α. anfractibus convexiusculis; sutura distincta.—f. 37. l. c.

Hab. in Maderâ et Portu S^{to}.

β. anfractibus planatis; sub-obsoleta.—f. 38. l. c.

Hab. in Maderâ.

Clausiliæ labiatæ Sow. (*C. bicanaliculatæ* sec. *Fer.*) nimis affinis; sed triplò minor; gracilior; anfractibus, etiam in β., minùs planis; sutura minùs obsoleta; plica posteriore aperturæ prominente, margini labri continua, nec interna; peristomate minùs incrassato, nec labroso. Variantur etiam et α. et β. collo aperturæ magìs minùsve producto. *Clausilia retusa* (*Bulimus retusus, Oliv.*) etiam forma et habitu magìs affinis; sed striolis exilissmis aliarum interstitia decussantibus differt.

70. Clausilia exigua, *Prodr. MS.*—Tab. 6. f. 39.

C. testa parvula, turrita, gracili, obtusa: anfractibus planiusculis, omnibus transversè creberrimè striatis: apertura obliquè obovata, biplicata; plica antica valde interna; postica prominente, margini peristomatis producta, continua, sinumque cum labro efficiente; ambabus columellaribus, simplicibus: peristomate continuo, reflexo, posticè subsinuato.

Long. 3—3½ lin. Diam. 1. Anfr. 8.

Hab. in Madera.

Clausiliæ parvulæ Leach, ut videtur, affinis.

3. *Familia, Cyclostomidæ.*

VII. GENUS, CYCLOSTOMA, *Lam^k.*

71. Cyclostoma lucidum, *Prodr. MS.*—Tab. 6. f. 40.

C. testa globoso-conoidea, nitida, læviuscula, sub-imperforata: anfractibus convexis, transversè sub-striatis; sutura impressa.

Axis 2 lin. Diam. 2½. Anfr. 5.

Hab. in Maderæ humidis sylvaticis.

Testa, quoad formam, *Valvatam piscinalem* referens, fusca, olivaceo-cornea, lucida.

TABULARUM EXPLICATIO.

Tab. I.

ff. 1. Plant of *Goodyera macrophylla, nob.* natural size.

f. 2. A single flower with its germen and bractea.

f. 3. The uppermost of the three outer petals of the perianth.

ff. 4. Two lowermost of ditto.

f. 5. The uppermost of the three outer and the two inner petals of the Perianth; the three cohering upwards by means of the former at the back of the two inner.

ff. 6. The same two inner petals, separated.

f. 7. Side view of f. 5.

f. 8. Labellum with column and anther-case, in situ.

f. 9. Side view of the same.

f. 10. Same as f. 9, with the anther-case lifted up (artificially) shewing the two Pollen-masses in situ.

f. 11. Inside of anther-case, shewing the dissepiment.

f. 12. Front view of the two Pollen-masses, in situ.

All the figures, except f. 1, more or less magnified.

Tab. II.

f. 1. Plant of *Tolpis crinita, nob.*; smoother and with more entire leaves than usual.

f. 2. The more common state of the same.

f. 3. A seed with its pappus; magnified.

Tab. III.

f. 1. Branch of *Sedum fusiforme, nob.*

f. 2. A single petal with its gland, and stamen; magnified.

Tab. IV.

Branch of *Ononis dentata, Sol.*

Tab. V.

f. 1. Vitrina Lamarckii, *nob.* { *a*, common state. *b*, state of the same in which the volutions are visible internally to the apex.

f. 2. Helix furva, *nob.* var. *a.*

f. 3. —— erubescens, *nob.* var. *a.*

f. 4. —— sub-plicata, *Sow.* [Testa ("viva" dicta) junior]

f. 5. —— undata, *nob.*

f. 6. —— phlebophora, *nob.*

f. 7. —— arcta, *nob.*—magnified.

f. 8. —— fausta, *nob.*—ditto.

f. 9. —— arridens, *nob.*—ditto.

f. 10. —— Webbiana, *nob.*—Two views of same individual.

f. 11. —— Bulveriana, *Wood.*—ditto, ditto.

f. 12. —— tectiformis, *Sow.*

f. 13. —— subtilis, *nob.*—magnified.

f. 14. —— actinophora, *nob.*

f. 15. —— Porto-Sanctana, *Sow.* var. *a. nob.*

f. 16. ——————, ——. — *β. nob.*

f. 17. —— pusilla, *nob.*—magnified.

f. 18. —— bifrons, *nob.*

f. 19. —— paupercula, *nob.*—magnified.

f. 20. —— obtecta, *nob.* { *a*, state in which it is found, coated with soil. *b*, the same cleaned.

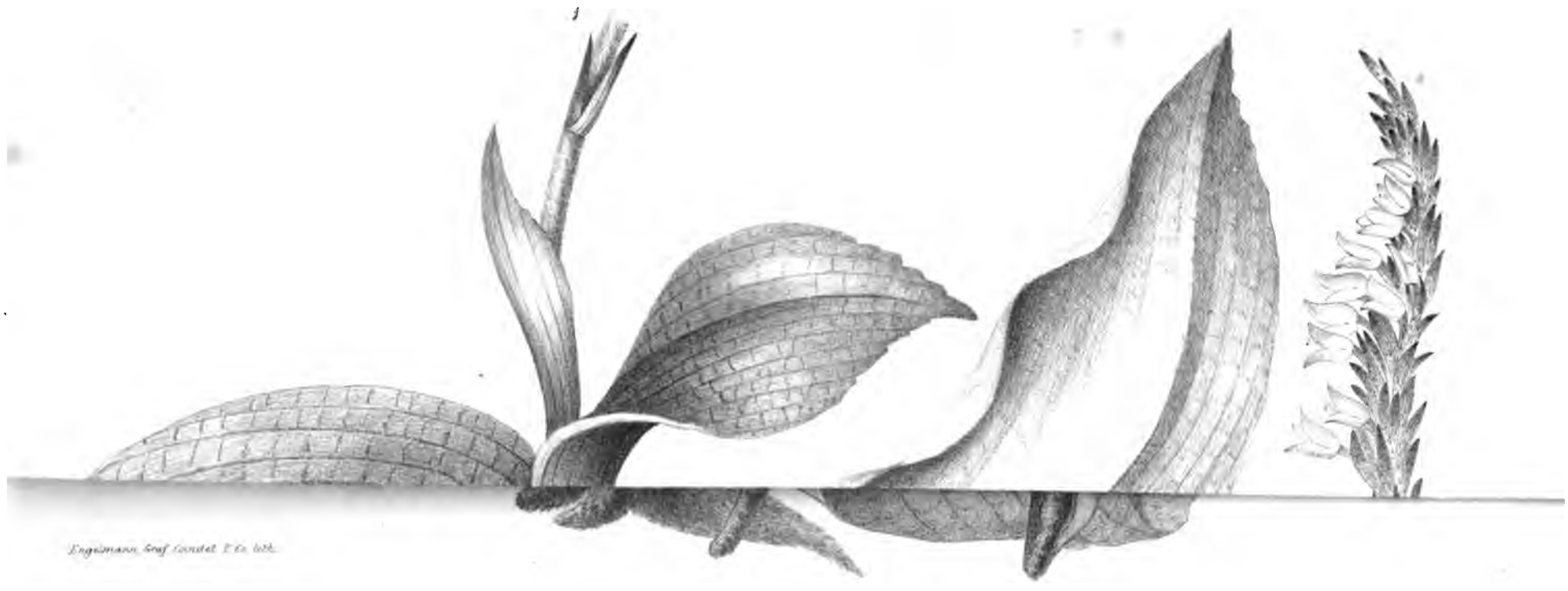
f. 21. —— dealbata, *nob.* var. *a. granulata.*

f. 22. Helix maderensis, *Wood.*
f. 23. —— compar, *nob.*—magnified.
f. 24. —— leptosticta, *nob.*—ditto.
f. 25. —— lentiginosa, *nob.*—ditto.
f. 26. —— calva, *nob.*

Tab. VI.

f. 1. Helix abjecta, *nob.*
f. 2. —— compacta, *nob.*
f. 3. —— consors, *nob.*
f. 4. —— depauperata, *nob.*
f. 5. —— lurida, *nob.*
f. 6. —— nitidiuscula, *Sow.*
f. 7. —— punctulata, *Sow.*, var. *a.* *nob.*
f. 8. —— ——, ——, — *β.* *nob.*
f. 9. —— lauta, *nob.*
f. 10. —— rotula, *nob.*
f. 11. —— polymorpha, *nob.* var. *a.* *irrasa.*
f. 12. —— ——, —. —. *β.* *depressiuscula.*
f. 13. —— ——, —. —. *γ.* *arenicola.*
f. 14. —— ——, —. —. *δ.* *attrita.*
f. 15. —— ——, —. —. *ε.* *calcigena.*
f. 16. —— ——, —. —. *ζ.* *pulvinata.*
f. 17. —— cheiranthicola, *nob.* Subvar. 1. *zonata.*
f. 18. —— oxytropis, *nob.*
f. 19. —— echinulata, *nob.*—magnified.
f. 20. —— duplicata, *nob.*—ditto.
f. 21. —— turricula, *nob.*
f. 22. —— bicolor, *nob.*—magnified.
f. 23. —— C. tornatellina, *nob.*—ditto.
f. 24. —— C. melampoides, *nob.*
f. 25. —— C. triticea, *nob.* var. *a.* *biplicata.*—magnified.

f. 26. Helix C. triticea, *nob.* var. β. edentula.—magnified.
f. 27. —— C. ovuliformis, *nob.*—ditto.
f. 28. —— C. gracilis, *nob.*—ditto.
f. 29. —— C. lubrica, *Mull.* var. *nob.*—ditto.
f. 30. —— C. anconostoma, *nob.* var. α.—ditto.
f. 31. —— C. cheilogona, *nob.*—ditto.
f. 32. —— C. sphinctostoma, *nob.*—ditto.
f. 33. —— C. monticola, *nob.*—ditto.
f. 34. —— C. calathiscus, *nob.*—ditto.
f. 35. —— C. cassida, *nob.*—ditto.
f. 36. Clausilia crispa, *nob.*—ditto.
f. 37. ——— deltostoma, *nob.* var. α.—ditto.
f. 38. ——— ———, *nob.* —. β.—ditto.
f. 39. ——— exigua, *nob.*—ditto.
f. 40. Cyclostoma lucidum, *nob.*—ditto.



Engelmann Graf Coindet & Co. lith.

T. 2

M. C. Berkeley del.

Engelmann sculp. Cornael & Co lith.

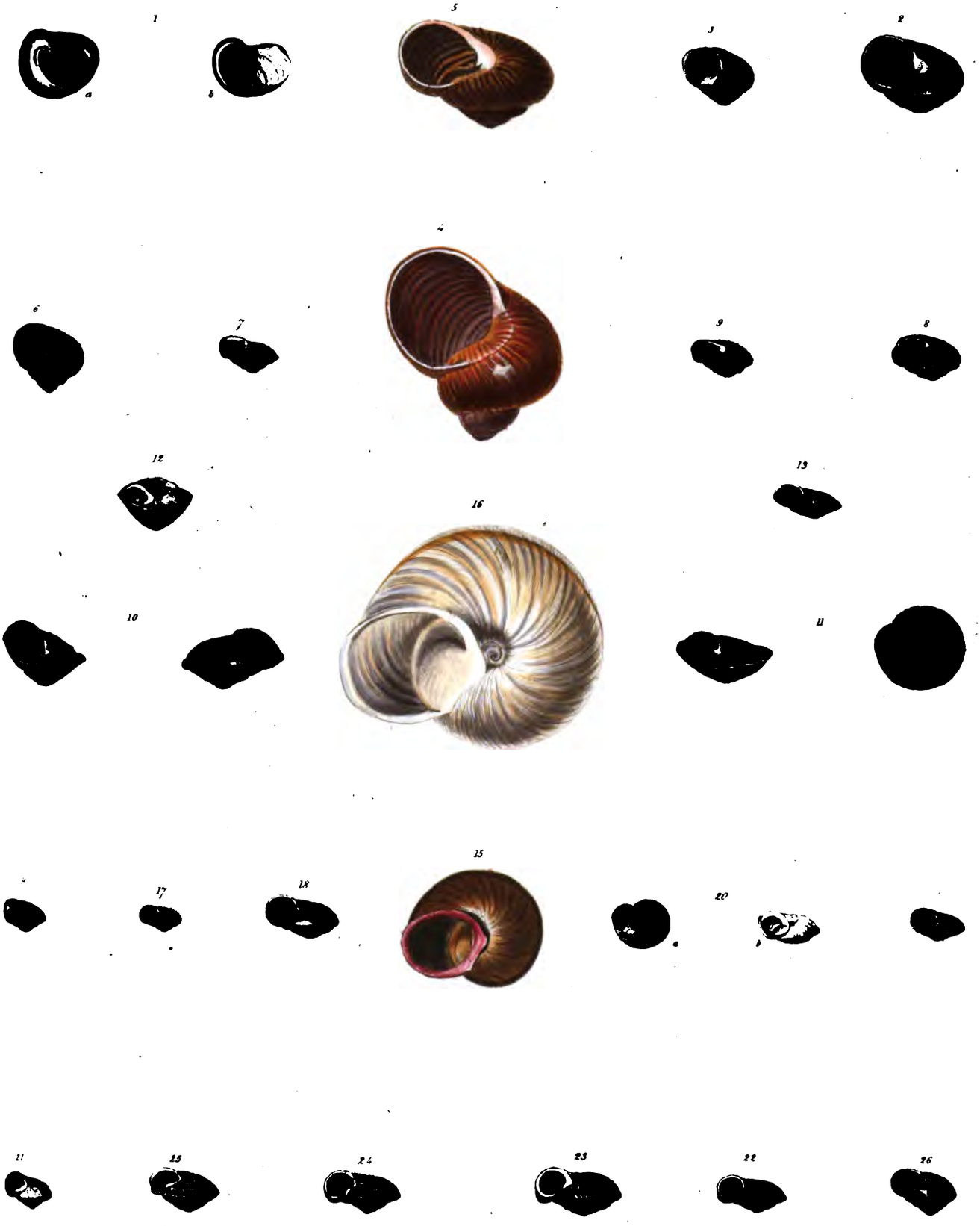
Tolpis crinita.

T. 3.

F. T. L. del.t

Engelmann. Graf, Coindet & Co lith.

Sedum fusiforme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II. *On the General Equation of Curves of the Second Degree.*

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[Read *November* 15, 1830.]

THE object of this Paper is to draw attention to some properties of Curves of the Second Degree, by means of which the reduction of their equations from one set of axes to another is materially facilitated. Little, if any, notice of these properties has been taken, nor do I remember to have seen their existence mentioned with the exception of two very limited particular cases, viz. that the sum of the squares of conjugate diameters, and the parallelogram formed by them are constant.

Suppose that any curve of the second degree is referred to axes which make an angle θ or that $\widehat{xy} = \theta$. Suppose the origin removed to a point whose co-ordinates are m and n in the directions of x and y, and moreover suppose the directions of the axes to be changed so that $\widehat{x'x} = \phi$ and $\widehat{y'x} = \psi$, x' and y' being the new co-ordinates; and let $\widehat{x'y'} = \psi - \phi = \theta'$. Let the equations of the curve referred to the first and second systems of axes be

$$a y^2 + b x y + c x^2 + d y + e x + f = 0, \qquad (1)$$

$$a' y'^2 + b' x' y' + c' x'^2 + d' y' + e' x' + f' = 0; \qquad (2)$$

From the suppositions made respecting the axes

$$x = \frac{\sin(\theta-\phi)}{\sin\theta}x' + \frac{\sin(\theta-\psi)}{\sin\theta}y' + m,$$
$$y = \frac{\sin\phi}{\sin\theta}x' + \frac{\sin\psi}{\sin\theta}y' + n. \qquad (3)$$

If we substitute in (1) these values of x and y, the resulting coefficients of y'^2, $x'y'$, &c. must be *proportional* to a', b', &c. Preserve this condition and make the substitutions and developments as follows:

Let
$$A = a - b\cos\theta + c\cos^2\theta,$$
$$B = \sin\theta(b - 2c\cos\theta),$$
$$C = c\sin^2\theta,$$
$$D = 2an + bm + d - \cos\theta(2cm + bn + e), \qquad (4)$$
$$E = \sin\theta(2cm + bn + e),$$
$$F = an^2 + bmn + cm^2 + dn + em + f.$$

In which it is important to observe that if the primitive co-ordinates be rectangular $A=a$, $B=b$ and $C=c$. If in addition the origin be not changed $D=d$, $E=e$, $F=f$. Also that if the axes of x' is parallel to that of x and $\widehat{x'y'} = 90°$, the equation becomes

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0.$$

The substitution above indicated will now give the following results:

$$\lambda a' \sin^2\theta = A\sin^2\psi + B\sin\psi\cos\psi + C\cos^2\psi,$$
$$\lambda b' \sin^2\theta = 2A\sin\psi\sin\phi + B(\sin\psi\cos\phi + \cos\psi\sin\phi) + 2C\cos\psi\cos\phi,$$
$$\lambda c' \sin^2\theta = A\sin^2\phi + B\sin\phi\cos\phi + C\cos^2\phi, \qquad (5)$$
$$\lambda d' \sin\theta = D\sin\psi + E\cos\psi,$$
$$\lambda e' \sin\theta = D\sin\phi + E\cos\phi,$$
$$\lambda f' = F,$$

where λ is any quantity whatever.

From which we find

$$\lambda^2(b'^2-4a'c')\sin^4\theta=(B^2-4AC)\sin^2(\psi-\phi)=(b^2-4ac)\sin^2\theta\sin^2\theta'$$

$$\text{or } \lambda^2\frac{b'^2-4a'c'}{\sin^2\theta'}=\frac{b^2-4ac}{\sin^2\theta} \qquad (6)$$

of which one particular case is that the parallelograms described about conjugate diameters are always the same.

Again we find that

$$\lambda\frac{a'+c'-b'\cos\theta'}{\sin^2\theta'}=\frac{a+c-b\cos\theta}{\sin^2\theta} \qquad (7)$$

of which when divided by (6) one particular case is that the sum of the squares of conjugate diameters is always the same.

When

$$a+c-b\cos\theta=0.$$

it indicates an equilateral hyperbola. Again, we find that

$$\lambda^3\frac{c'd'^2+a'e'^2-b'd'e'}{\sin^2\theta'}=\frac{cd^2+ae^2-bde-(b^2-4ac)(an^2+bmn+cm^2+dn+em)}{\sin^2\theta} \qquad (8)$$

from (8), (6) and the last of (5) we deduce that

$$\lambda\left(\frac{c'd'^2+a'e'^2-b'd'e'}{b'^2-4a'c'}+f'\right)=\frac{cd^2+ae^2-bde}{b^2-4ac}+f \qquad (9)$$

If each side of this be nothing, the equation represents, either two straight lines which intersect, a point, or is impossible.

If the new co-ordinates be such that $b'=0$ we find from the second of (5)

$$2A\tan\phi\tan\psi+B(\tan\phi+\tan\psi)+2C=0.$$

If in addition to this the new axes must be rectangular, in which case $\tan\phi\tan\psi+1=0$, we find the following equation

$$\tan 2\chi=\frac{B}{C-A}=-\frac{\sin\theta\{b-2c\cos\theta\}}{a-b\cos\theta+c(\cos^2\theta-\sin^2\theta)} \qquad (10)$$

from which two values of χ are found differing by 90°, either of which may be the value of ϕ, and the other of ψ.

The lines determined by these angles are parallel to the principal diameters. The next question is, how can the greater principal diameter be distinguished from the lesser? It must be observed that in the general equation

$$\frac{a}{c} = \frac{\text{rectangle of segments of the axis of } x}{\text{rectangle of segments of the axis of } y} = \frac{(\text{diameter parallel to axis of } x)^2}{(\text{diameter parallel to axis of } y)^2}.$$

Since the square of one or both of these diameters may be negative, that which is ***numerically*** the greater can be ascertained only when we know the sign of c^2-a^2. If this be positive the greatest of the two diameters above-mentioned is parallel to the axis of y, &c.

In (10) one of the values of χ must be less than 90°. Take this for the new axis of x, that is, let $\sin 2\phi$ be positive. From (10) it appears that

$$\sin 2\phi = \frac{B}{M} \quad \cos 2\phi = \frac{C-A}{M} \qquad (11)$$

where $M = \pm\sqrt{B^2+(A-C)^2}$, and since $\sin 2\phi$ is supposed positive, M must be of the same sign as B. Also $\sin\psi = \cos\phi$, $\cos\psi = -\sin\phi$, since $\psi-\phi = 90°$. Therefore from (5)

$$\lambda a' \sin^2\theta = A\cos^2\phi - B\sin\phi\cos\phi + C\sin^2\phi,$$

$$\lambda c' \sin^2\theta = A\sin^2\phi + B\sin\phi\cos\phi + C\cos^2\phi,$$

$$\lambda(c'+a')\sin^2\theta = A + C,$$

$$\lambda(c'-a')\sin^2\theta = (C-A)\cos 2\phi + B\sin 2\phi = M \text{ from (11)}.$$

Therefore $c'^2-a'^2$ has the same sign as $M(A+C)$

..... $B(A+C)$,

since B and M have the same signs; the hypothesis being that the principal diameter which is nearest to the axis of x in the positive direction is parallel to the axis of x'. Accordingly therefore as $B(A+C)$ or $\sin\theta\,(b-2c\cos\theta)(a+c-b\cos\theta)$ is positive or

negative, the less or greater principal diameter is nearer to the axis of x.

When $B(A+C)=0$ there are two cases to be distinguished. If $A+C=0$ the curve is equilateral, but can only be an hyperbola. If $B=0$ and $C-A$ is not $=0$ from (11), one of the principal diameters is parallel to the axis of x. Which it is may be determined from the sign of c^2-a^2.

When $B=0$ and $C-A=0$, or which is the same, $b=2c\cos\theta$ and $a=c$, the position of the principal diameters is indefinite; that is, the curve is a circle.

To determine the magnitude of the principal diameters, the equation must be reduced to the form

$$\lambda a'y^2+\lambda c'x^2+\lambda f'=0, \text{ where } \theta'=90^0.$$

In this case the equations (6), (7) and (9) become

$$-4\lambda^2 a'c'=\frac{b^2-4ac}{\sin^2\theta}$$

$$\lambda a'+\lambda c'=\frac{a+c-b\cos\theta}{\sin^2\theta}$$

$$\lambda f'=\frac{cd^2+ae^2-bde}{b^2-4ac}+f.$$

Whence the squares of the principal semidiameters or $-\frac{f''}{a'}$ and $-\frac{f'}{c'}$ are contained in the formula

$$-2\sin^2\theta\frac{\frac{cd^2+ae^2-bde}{b^2-4ac}+f}{a+c-b\cos\theta\pm\sqrt{(a-c)^2+b^2-2\cos\theta(a+c)b-2ac\cos\theta)}}.$$

In this way the curve might be referred to the conjugate diameters which make a given angle θ', and the limits of the value of θ' might be determined.

When the asymptotes are the axes of co-ordinates, the equation of the hyperbola is $\lambda b' x'y' + \lambda f' = 0$.

The position of the asymptotes is then determined from the equation

$$A \tan^2 \chi + B \tan \chi + C = 0,$$

$$\text{or } \tan \chi = \frac{\sin \theta}{2} \cdot \frac{b - 2c \cos \theta \pm \sqrt{b^2 - 4ac}}{a - b \cos \theta + c \cos^2 \theta},$$

where the two values of χ are those of ϕ and ψ.

The equations (6), (7) and (9) become

$$\frac{\lambda^2 b'^2}{\sin^2 \theta'} = \frac{b^2 - 4ac}{\sin^2 \theta}$$

$$-\frac{\lambda b' \cos \theta'}{\sin^2 \theta'} = \frac{a + c - b \cos \theta}{\sin^2 \theta}$$

$$\text{and } \lambda f' = \frac{cd^2 + ae^2 - bde}{b^2 - 4ac} + f$$

$$\text{whence } \tan \theta' = \pm \frac{\sin \theta \sqrt{b^2 - 4ac}}{a + c - b \cos \theta}$$

and the equation is

$$\pm \frac{b^2 - 4ac}{\sqrt{(a-c)^2 + b^2 - 2\cos\theta(a+c)\, b - 2ac\cos\theta)}} x'y' + \frac{cd^2 + ae^2 - bde}{b^2 - 4ac} + f = 0.$$

In referring to the expression for the principal semidiameters, it appears that if $a + c - b \cos \theta$ has the same sign as $\frac{cd^2 + ae^2 - bde}{b^2 - 4ac} + f$, the greater diameter is possible, or the curve lies in the acute angles of the asymptotes, and the contrary.

If the equation of the parabola be reduced to the form $a'y'^2 + e'x' = 0$, where $\theta = 90^\circ$, the equations (7) and (8) take the following form

$$\lambda a' = \frac{a + c - b \cos \theta}{\sin^2 \theta} = \frac{a - 2\sqrt{ac} \cos \theta + c}{\sin^2 \theta}$$

$$\lambda^3 a' e'^2 = \frac{cd^2 + ae^2 - bde}{\sin^2\theta} = \frac{(\sqrt{c}\,d - \sqrt{a}\,e)^2}{\sin^2\theta}$$

whence the equation becomes

$$y^2 \pm \frac{\sqrt{c}\,d - \sqrt{a}\,e}{(a - 2\sqrt{ac}\cos\theta + c)^{\frac{1}{2}}} \sin^2\theta x = 0.$$

The position of the axis is determined by the same criterion as that for finding the major axis of the ellipse; and the curve may be further ascertained if we recollect that it must lie entirely on one side of the right line whose equation is $dy + ex + f = 0$, and on that side in which the co-ordinates of a point are such as to make $dy + ex + f$ of a different sign from a or c.

In order to find the co-ordinates of the vertex, we must have recourse to equations (5) recollecting that $B^2 - 4AC = 0$, and that $\tan\phi \tan\psi + 1 = 0$. From the third and fourth of these we find

$$\frac{E}{D} = -\sqrt{\frac{A}{C}} = -\frac{\sqrt{a} - \sqrt{c}\cos\theta}{\sqrt{c}\sin\theta}.$$

But the fourth and fifth of (4) give the following equation

$$\frac{D}{E} = \frac{2\sqrt{a}\,k + d - \cos\theta(2\sqrt{c}\,k + e)}{(2\sqrt{c}\,k + e)\sin\theta}$$

where $k = \sqrt{a}\,n + \sqrt{c}\,m$

and the last of (4) gives

$$k^2 + dn + em + f = 0,$$

from the two first of which we find

$$k = \sqrt{a}\,n + \sqrt{c}\,m = \tfrac{1}{2}\frac{(e\sqrt{a} + d\sqrt{c})\cos\theta - (d\sqrt{a} + e\sqrt{c})}{a - 2\sqrt{ac}\cos\theta + c},$$

and from the last

$$dn + em = -f - k^2,$$

from which equations m and n may be found. The expressions present nothing remarkable.

The consequences of these formulæ might be carried further, but what has been here said may be sufficient to turn the attention of elementary writers to this part of the subject. Analogous formulæ may be obtained for Surfaces of the Second Degree, which will probably form the subject of a future communication.

AUGUSTUS DE MORGAN.

III. *On the Nature of the Light in the Two Rays produced by the Double Refraction of Quartz.*

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I PROPOSE in this paper to offer some conjectures as to the nature of the light forming the two rays produced by the double refraction of quartz; to describe the experiments on which they are founded; and to explain the calculations by which the theory and the experiments are compared. The subject is one to which (I believe) no attention has been paid, except by one distinguished foreigner; the mode of calculation is original to me, and is, to the best of my knowledge, new.

It is well known that the rays produced by the double refraction of calc spar, (calcareous spar, Iceland spar, or rhombohedral carbonate of lime) and most other doubly refracting crystals, are entirely polarized: one in the principal plane passing through the ray (or, if a biaxal crystal, in the plane equally inclined to the planes passing through the ray and the two axes) and the other in a plane perpendicular to the former. From the exact agreement of the phenomena of depolarization with the calculations made on this hypothesis, we are justified in supposing that the law holds true when the rays are so little

separated that it is difficult to observe them in the common mode of inspection. Now it has generally been supposed that the two rays of quartz are polarized in the same way: differing from those of calc spar only in the magnitude and direction of their separation. It was known however almost as soon as Arago and Biot commenced their observations, that there is some anomaly in the rays passing in the direction of the axis of quartz; and the latter of these observers established the difference of right-handed and left-handed quartz. Fresnel by a simple experiment* (which I have repeated) shewed that the light in the direction of the axis of quartz is not one ray, but two rays moving in the same direction, and with different velocities. He shewed moreover that a new kind of light may be produced by causing polarized light to undergo two internal reflections in a glass rhomb with certain angles, the plane of polarization being inclined to the plane of incidence at an angle of 45°; and that this light is exactly similar to one or other of the two rays abovementioned according as the plane of polarization is on one or the other side of the plane of incidence. And by a mathematical investigation, of which I am unable to supply the deficient steps, he shewed that the effect of the internal reflections is to retard by one quarter of an undulation the undulations perpendicular to the plane of incidence, so that in the light thus modified the particles of ether which were originally in a straight line will at any time be found in the form of a circular helix, and each will revolve uniformly in a circle†. And from the nature of the original experiment

* It is not easy to make this experiment in a satisfactory manner. If the axes of the crystals are not precisely adjusted, several images will be seen. I have not succeeded in obtaining two only, though I have made the others much more faint than the two principal.

† As I cannot appreciate the mathematical evidence for the nature of circular polarization, I shall mention the experimental evidence on which I receive it. 1st. The light when received

it appeared that in right-handed quartz it is necessary to suppose the right-circular polarization transmitted with the greater velocity; in left-handed quartz the contrary. I have repeated and varied most of Fresnel's experiments relating to this subject, and am perfectly convinced of the correctness of his views.

Now if, in the experiment with the glass rhomb, the planes of polarization and incidence be inclined at any other angle than 45°, the magnitudes of the undulations parallel and perpendicular to the plane of incidence will no longer be equal: but the alteration of their periods will be the same as before. The displacement of the particles of ether will still be represented by a helix, but instead of being traced round a circular cylinder, it must be supposed traced round an elliptic cylinder. This modification may properly be called (as Fresnel has called it) elliptical polarization*. This term has since been used by Dr. Brewster to express the nature of the light (probably identical with this, or nearly so) reflected from metallic surfaces.

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ceived on an analyzing plate or tourmaline presents the same appearance in whatever direction the analyzing plate is turned round the incident ray. 2d. The phenomena of depolarization are the same in whatever direction the analyzing plate is turned. 3d. If the polarized light passes through two such rhombs placed in similar positions, the plane of polarization is shifted 90°. 4th. If they are placed in crossed portions the plane of polarization is unaltered. 5th. The phenomena of depolarization agree with the calculations founded on this supposition: in uniaxal crystals, where the plane of polarization of one ray is changed 360° in going round the axis, the alternate quadrants are pushed in and thrust out one quarter of a tint: and in biaxal crystals, where the plane of polarization is changed only 180° in going round the axis, the alternate semicircles are altered in the same manner.

* If I might venture to fix on the discovery of Fresnel, which among all his wonderful additions to optical science appears likely to possess the greatest practical value, I should select his invention of the mode of producing circularly-polarized or elliptically-polarized light by internal reflexion of plane-polarized light in glass or water. He has given us the power of producing light whose laws are as well known as those of plane-polarized light, and which is more manageable, inasmuch as it admits of degrees in its ellipticity. The beautiful geometry of Malus is forgotten when we think of the discovery of polarization: the far more valuable theoretical discoveries of Fresnel will lose their preeminence when put in competition with an invention which enables others to make discoveries.

Should any difference be found, I have no hesitation in fixing on the modification above described as that to which it ought in propriety to be attached.

I am now able to explain my conjectures on the nature of the light in the two rays of quartz.

1. I suppose the ordinary ray to consist of light elliptically polarized, the greater axis of the ellipse being perpendicular to the principal plane; and the extraordinary ray to consist of light elliptically polarized, the greater axis of the ellipse being in the principal plane.

2. I suppose that when the ordinary ray is right-elliptically-polarized, the extraordinary ray is left-elliptically-polarized: and *vice versâ*.

3. I suppose that the proportions of the axes of the two ellipses are the same: each proportion being one of equality when the direction of the ray coincides with the axis, and becoming more unequal, according to some unknown law, as the direction is more inclined to the axis: the minor axes of the ellipses having sensible magnitudes when the rays are inclined 10° to the axis.

4. I suppose that the course of the rays after refraction can be determined by the construction given by Huyghens for calc spar, with this difference only, that the prolate spheroid for determining the course of the extraordinary ray must not be supposed to touch the sphere for determining the course of the ordinary ray, but must be entirely contained within it.

These conjectures were originally suggested by the desire of finding some connecting link between the peculiar double refraction in the axis discovered by Fresnel*, and the double

* It does not appear, I think, that Fresnel had made any distinct supposition as to whether the two rays in the axis should be considered as the ordinary and extraordinary ray in their ultimate

refraction commonly recognized. All the phenomena of colours which I have observed agree perfectly with the results of my hypotheses.

I may mention that I have found observations corresponding to many detached parts of the phenomena which I have viewed assembled, in the early memoirs of Arago and Biot. But the method which these philosophers used, (particularly the latter), of examining a small part only at a time, does not appear to be well adapted to the discovery of the laws of light. In the experiments which I am about to describe, every thing depends on the form of the coloured curves; and to attempt to discover this from observation of detached parts would be perfectly hopeless. These coincidences I have recognized only since I made my own observations.

It must be observed that all the phenomena mentioned below are described as they appear when examined with an analyzing plate of unsilvered glass. If a plate of tourmaline be used, the right and left parts of the image will have the same relative position, but the upper and lower will be interchanged: the observer's eye being supposed to turn in such a manner that the axis of the tourmaline appears *up* and *down*.

PHENOMENA.

I. If a plate of calc spar cut perpendicular to the axis be examined with the polarizing and analyzing plates crossed, the system of rings is that represented in fig. 1. If the analyzing plate be turned less than 90° either way round the incident ray, the system of rings is that represented in fig. 2: and if turned

ultimate state, or not. From all that I could extract from his Memoirs, I was always in doubt whether both the ordinary and the extraordinary ray in the neighbourhood of the axis ought not to be considered as divided each into two circularly polarised rays.

exactly 90°, it is that of fig. 3. The order of colours does not sensibly differ from Newton's scale, beginning with black. These are common and well known phenomena.

II. If Fresnel's rhomb of glass, mounted as represented in fig. 4, be placed to receive the polarized light, so that the plane of reflection pass through the divisions 45° and 225°, the calc spar will present the appearance of fig. 5. The rings are abruptly and absolutely dislocated: those in the upper right-hand quadrant and the quadrant opposite to it are pushed from the center by one-fourth of an interval, and those in the other quadrants are drawn nearer to the center by the same quantity. The line separating the quadrants is no-where black: the intensity of its light is uniform and about equal to the mean intensity. If the plane of incidence pass through 135° and 315°, the phenomena of adjacent quadrants are exactly interchanged. No alteration is made by turning the analyzing plate round the incident ray: the lines dividing the quadrants are always parallel and perpendicular to the plane of reflexion at the analyzing plate*.

III. If the plane of reflexion in the rhomb pass through 0° and 180°, or through 90° and 360°, the phenomena are precisely the same, and undergo the same changes as those in Phenomenon I. If while the plates are crossed the rhomb be turned gradually from the position 0° towards 45°, the rings are gradually changed, at first becoming (as far as the eye can judge) elliptical, and then assuming the form represented in fig. 6.

IV. If a plate of quartz, whether right or left-handed, be

* It is proper to mention that I had exhibited this phenomenon to the Cambridge Philosophical Society in the spring of last year, long before the publication of Dr. Brewster's valuable Memoir in the Phil. Trans. for 1830, and (I believe, but I do not recollect the date) before its communication to the Royal Society.

interposed between the crossed plates, a set of rings is seen as in figs 7, 8, 9, 10. As far as the eye can judge, the rings are exactly circular, but there is no black cross, and the central tint is not black, but removed from it by a number of tints in Newton's scale proportional to the thickness of the quartz. Thus with a thickness 0,48 inch, the central tint is pale pink: with a thickness 0,38 inch, the central tint is bright yellowish green: with thickness 0,26 inch, it is a rich red plum-colour: with thickness 0,17 inch: it is a rich yellow.

The colours then appear to be nearly the same, beginning from the center, as in Newton's scale, beginning with the tint representing this central tint. At a considerable distance from the center four dark brushes begin to be visible, in the same directions as the arms of the black cross in calc spar.

V. Now (supposing the crystal right-handed), if the plate of quartz be thin, and the analyzing plate be turned, the upper part towards the observer's left hand, a blueish short-armed cross appears in the center*, which on turning further becomes yellow: and the rings are enlarged. On turning still further, the cross breaks into four dots. The rings are no longer circular, but of a form intermediate between a circle and a square, their diagonals (as well as the cross) being inclined to the left of the parallel and perpendicular to the plane of reflexion. See fig. 11. If the analyzing plate be turned the other way, there is no cross: the form of the rings is changed from circular nearly as in the former case.

VI. If the plate of quartz be thick, the dilatation of the rings and the change of form are all the perceptible phenomena.

* This may be considered as the definition of *right-handedness* of the crystal: and this observation gives the readiest means, with a thin plate, of determining whether it is right-handed or left-handed. If the plate be thick, the easiest method is to observe in which direction the analyzing plate must be turned to make the rings dilate.

And on turning the analyzing plate continually to the left, the rings continually dilate, and new spots start up continually in the centre, and become rings. If the crystal be left-handed, the remarks in this and the last article apply equally well, supposing the analyzing plate turned in the opposite direction.

VII. If Fresnel's rhomb be placed in the position 45°, and the light thus circularly polarized pass through the quartz; on applying the analyzing plate, instead of rings there are seen two spirals mutually inwrapping each other as in fig. 12. If the rhomb be placed in position 135°, the figure is turned through a quadrant. If the quartz be left-handed, the spirals are turned in the opposite direction. The central tint appears to be white. With the rhomb which I have commonly used (which is of plate glass, but with the angles given by Fresnel for crown glass) there is at the center an extremely dilute tint of pink: I think it likely that this arises from the error in the angles, as the intensity of the colour bears no proportion to that in other parts of the spirals. The figure was drawn from the appearances given by a plate of quartz 0,26 inch thick.

VIII. If two plates of quartz of equal thickness, but cut one from a right-handed and the other from a left-handed crystal, be attached together, and put between the polarizing and analyzing plates, the left-handed slice nearest to the polarizing plate, the appearance presented is that of fig. 13. Four spirals (proceeding from a black cross in the center, which is inclined to the plane of reflexion) cut a series of circles at every quadrant. The points of intersection are in the plane of reflexion, and perpendicular to it. This is the simplest way of describing the form: but if we followed the colours which graduate most gently, we should say that the form of each is alternately a spiral and circular arc, quadrant after quadrant.

At a distance from the center the black brushes are seen. If the combination be turned so that the right-handed slice is nearest to the polarizing plate, the spirals are turned in the opposite direction. This is one of the most beautiful phænomena of optics. The slices from whose appearance the figure was drawn are each 0,16 inch thick.

I shall now proceed to explain the mode of calculating these phænomena on assumed laws of the nature of light in the two rays of crystals.

In fig. 14, let AB, CD, be two parallel rays of the same pencil incident on a plate of calc spar cut perpendicular to its axis, of which one furnishes the ordinary ray BE, and the other the extraordinary ray DE, which afterwards pass in the same direction EF. (The extraordinary ray of AB, and the ordinary ray of CD are not to be considered here, as they do not emerge at E: but each of them will interfere with some other ray). The paths are found by this construction. Draw GK a tangent to a circle whose radius is $GH \times b$ (preserving Biot's notation); draw LN a tangent to the ellipse whose semi-axes are $LM \times a$, $LM \times b$. The velocity and direction of the ray will be represented by the radius joining the point of incidence with the point of contact, the velocity in air being represented by GH, LM. Hence the path of the ordinary ray (measured by the path in air which it would have described in the same time) exceeds that of the extraordinary by

$$OB + \frac{GH}{BK} \cdot BE - \frac{LM}{DN} \cdot DE.$$

Putting θ for the angle of incidence, and T for the thickness of the plate, this is found (after all reductions)

$$= \frac{T}{b} \{\sqrt{1-b^2 \sin^2\theta} - \sqrt{1-a^2 \sin^2\theta}\}.$$

When θ is small, this is nearly $= T \times \frac{a^2-b^2}{2b} \times \theta^2$. Call this Θ.

I shall suppose with Fresnel, that by a ray polarized in one plane is meant a ray whose vibrations are entirely perpendicular to that plane; that consequently the vibrations forming the ordinary ray in the crystal are entirely perpendicular to the principal plane passing through that ray, and that those forming the extraordinary ray are wholly parallel to that plane.

I*. Now suppose a pencil of polarized light to fall with a small angle of incidence on a plate of calc spar cut perpendicular to its axis. Let us conceive ourselves looking in the direction of the incident ray; and let fig. 15. represent the projections (on a plane perpendicular to the incident ray) of the planes of polarization of the polarizing and analyzing plate, and of the principal plane of the crystal passing through the incident ray. Let P_2Ap_2 the plane of polarization (or of reflection) at the analyzing plate, make an angle α with P_1Ap_1 the original plane of polarization; and let CAc the principal plane of the crystal passing through the ray make the angle ϕ with the former plane. The displacement of the particles of ether produced by the wave as originally polarized, may be represented by $c.\sin\frac{2\pi}{\lambda}(vt-x)$, where λ is the interval of space between two waves, x the distance measured from any arbitrary point, t the time since the ether at that point was at rest, and v the velocity of the wave. (This applies even after the wave has passed through any media, provided we take for x the space which would in the same time have been described in air). And this displacement is entirely perpendicular to P_1p_1. This may be resolved into $c.\sin\frac{2\pi}{\lambda}.(vt-x).\cos(\alpha+\phi)$ perpendicular to AC, and

* The Phænomena, and the investigations corresponding to them, are numbered in the same way.

$c \cdot \sin \overline{\frac{2\pi}{\lambda}(vt-x)} \cdot \sin(\alpha+\phi)$ parallel to AC. The former of these furnishes the ordinary ray, the latter the extraordinary. Now after they have passed through the crystal, we may still keep the expression for the vibration of the ordinary ray, provided we make the proper alteration in the value of x: but (by what has gone before) we must then suppose the path of the extraordinary ray shorter by Θ. Consequently after passing through the crystal, the vibration produced by the ordinary ray is

$$c \cdot \sin \overline{\frac{2\pi}{\lambda}(vt-x)} \cdot \cos \overline{\alpha+\phi} \text{ perpendicular to } AC,$$

and that produced by the extraordinary ray is

$$c \cdot \sin \overline{\frac{2\pi}{\lambda}(vt-x+\Theta)} \cdot \sin \overline{\alpha+\phi} \text{ parallel to } AC.$$

When these are received on the analyzing plate, those parts only are transmitted to the eye or to the screen, which are perpendicular to AP_2. They are

$$c \cdot \sin \overline{\frac{2\pi}{\lambda}(vt-x)} \cdot \cos \overline{\alpha+\phi} \cdot \cos \phi,$$

$$\text{and } c \cdot \sin \overline{\frac{2\pi}{\lambda}(vt-x+\Theta)} \cdot \sin \overline{\alpha+\phi} \cdot \sin \phi:$$

and the sum of these represents the magnitude of the vibration which comes to the eye, or falls on the screen. It may be put under this form

$$\sin \overline{\frac{2\pi}{\lambda}(vt-x)} \left\{c \cdot \cos \overline{\alpha+\phi} \cdot \cos \phi + c \cdot \cos \frac{2\pi}{\lambda}\Theta \cdot \sin \overline{\alpha+\phi} \cdot \sin \phi\right\}$$

$$+ \cos \overline{\frac{2\pi}{\lambda}(vt-x)} \cdot c \cdot \sin \frac{2\pi}{\lambda}\Theta \cdot \sin \overline{\alpha+\phi} \cdot \sin \phi.$$

Now it must be remarked (as a general theorem which we shall use hereafter without further explanation) that an expression of the form

$$E.\sin\overline{\frac{2\pi}{\lambda}(vt-x)}+F.\cos\overline{\frac{2\pi}{\lambda}(vt-x)}$$

may always be put under the form

$$\sqrt{E^2+F^2}.\sin\overline{\frac{2\pi}{\lambda}\left(vt-x+\frac{\lambda G}{2\pi}\right)},$$

where $\tan G=\frac{F}{E}$, and G is constant for that ray. It is plain that this expresses a periodical vibration similar to that which we have all along supposed, and whose coefficient instead of c is $\sqrt{E^2+F^2}$. It is convenient to take the square of this coefficient as the measure of the intensity of light: and thus E^2+F^2 will represent the intensity in all cases similar to that before us.

In the present instance, the intensity or the sum of the squares of the multipliers of the sine and cosine of $\overline{\frac{2\pi}{\lambda}(vt-x)}$ is

$$c^2\{\cos^2\overline{\alpha+\phi}.\cos^2\phi+\sin^2\overline{\alpha+\phi}.\sin^2\phi$$

$$+2\cos\frac{2\pi}{\lambda}\Theta.\sin\overline{\alpha+\phi}.\cos\overline{\alpha+\phi}.\sin\phi.\cos\phi\}$$

$$=\frac{c^2}{2}\{1+\cos 2.\overline{\alpha+\phi}.\cos 2\phi+\cos\frac{2\pi}{\lambda}\Theta.\sin 2.\overline{\alpha+\phi}.\sin 2\phi\}.$$

Thus we have a general expression for the intensity of the light when polarized light passes in any one direction through the crystal, and after being reflected by the analyzing plate, is received on a screen. If we suppose polarized light to fall in all possible directions (within certain limits) upon the crystal, we must give all possible values to θ and ϕ, and we shall have the

intensity of light on all the different parts of the screen. It will be remarked that θ is very nearly proportional to the radius vector of the corresponding point on the screen, and ϕ the angle measured from the lower part towards the right.

If the eye be placed to receive the light coming from different parts of the analyzing plate, the appearance will be reversed with regard to right and left, and here the angle ϕ must be measured from the lower point towards the left.

1st. Let the plates be crossed, or $\alpha = 90^0$:

$$\cos 2.\overline{\alpha+\phi} = -\cos 2\phi: \quad \sin 2.\overline{\alpha+\phi} = -\sin 2\phi:$$

and the intensity is

$$\frac{c^2}{2}.\sin^2 2\phi.\sin^2\frac{\pi\Theta}{\lambda}:$$

or putting for Θ its value, the intensity is

$$\frac{c^2}{2}.\sin^2 2\phi.\sin^2\left(\frac{T\pi}{\lambda}.\frac{a^2-b^2}{2b}.\theta^2\right).$$

This is 0, or there is darkness, if $\phi = 0$, or $= 90^0$, or $= 180^0$, or $= 270^0$, whatever be the value of θ. This shews that there is a black cross through the center, parallel and perpendicular to the plane of reflection. Also there is darkness whatever be the value of ϕ, if

$$\frac{T\pi}{\lambda}.\frac{a^2-b^2}{2b}.\theta^2 = 0, \ = \pi, \ = 2\pi, \text{ \&c.},$$

$$\text{or if } \theta^2 = 0, \ = \frac{2b\lambda}{T(a^2-b^2)}, \ = \frac{4b\lambda}{T(a^2-b^2)}, \text{ \&c.}$$

This shews that there is a dark spot at the center, and a succession of dark rings, of which the difference between the radii diminishes continually. Cæteris paribus, the squares of the diameters of these rings are inversely as $\frac{a^2-b^2}{b}$, or are least in the crystal where the double refrac-

tion is greatest; and are inversely as T, or are least when the thickness of the plate is greatest. They are directly as λ, and consequently are greater for red rays than for blue. Hence after a little time the bright rings of one colour correspond with the dark of another. This gives the peculiar coloured character to the rings: it also prevents any of them from being totally black: whereas the evanescence of light in the cross is independent of λ, and the cross is totally black.

2^{d}. Let the plates be parallel or opposite: or $\alpha = 0$. The expression for the intensity becomes

$$\frac{c^2}{2}\{1+\cos^2 2\phi+\cos\frac{2\pi}{\lambda}\Theta\,.\sin^2 2\phi\}.$$

If $\phi = 0$, or $= 90^0$, or $= 180^0$, or $= 270^0$, this becomes c^2: thus there is a bright cross instead of a dark one. For other values of ϕ the light is greatest if $\frac{2\pi}{\lambda}\Theta = 0$, or $= 2\pi$, &c., and least if $\frac{2\pi}{\lambda}\Theta = \pi$, or $= 3\pi$, &c.: in the former case it $= c^2$, in the latter $c^2\cos^2 2\phi$. These indications point out exactly the form of fig. 2. In fact it is easily seen from the expressions that the intensities in corresponding parts of fig. 1. and fig. 2. are precisely complemental.

3rd. In the general case, if $\sin 2\phi = 0$ (that is if $\phi = 0$, or $= 90^0$, or $= 180^0$, or $= 270^0$) the expression becomes

$$\frac{c^2}{2}\{1+\cos 2\,.\overline{\alpha+\phi}\,.\cos 2\phi\} = \frac{c^2}{2}\{1+\cos 2\alpha\}.$$

This indicates a faint cross, which is bright when α is small, and dark when α is nearly $= 90^0$. And if $\sin 2\,.\overline{\alpha+\phi} = 0$, another cross of equal intensity is found, inclined to the former at an angle α. Generally if ϕ be between 0 and $90^0 - \alpha$, the intensity is greatest when $\frac{2\pi}{\lambda}\Theta = 0$, $= 2\pi$, &c., and least when $\frac{2\pi}{\lambda}\Theta = \pi$, $= 3\pi$, &c.: but if ϕ be between $90^0 - \alpha$ and 90^0, the intensity is greatest when $\frac{2\pi}{\lambda}\Theta = \pi$, $= 3\pi$, &c., and

least when $it = 0$, $= 2\pi$, &c.: and the same holds if we increase all these angles by 90°. Thus there is in fig. 3. a mixture of parts of the two systems of rings in figs 1. and 2.

II. Now let a Fresnel's rhomb be interposed, and let RAr, fig. 16. represent the plane perpendicular to the plane of internal reflection, and making the angle β with the plane of original polarization; the rest as before. The vibration $c \cdot \sin \overline{\frac{2\pi}{\lambda}(vt - x)}$ perpendicular to P_1A may be resolved into

$$c \cdot \sin \overline{\frac{2\pi}{\lambda}(vt-x)} \cdot \sin\beta \text{ parallel to } Ar,$$

$$\text{and } c \cdot \sin \overline{\frac{2\pi}{\lambda}(vt-x)} \cdot \cos\beta \text{ perpendicular to } Ar$$

Of these the former, by Fresnel's theory, is retarded one quarter of an undulation, or the latter is accelerated as much. Adopting the latter supposition, we must suppose that after the emersion from the rhomb, the vibrations are

$$c \cdot \sin \overline{\frac{2\pi}{\lambda}(vt-x)} \cdot \sin\beta \text{ parallel to } Ar,$$

$$\text{and } c \cdot \sin \overline{\frac{2\pi}{\lambda}(vt-x) + 90^\circ} \cdot \cos\beta, \text{ or } c \cdot \cos \overline{\frac{2\pi}{\lambda}(vt-x)} \cdot \cos\beta$$

perpendicular to Ar. These will furnish for the ordinary ray of the crystal

$$c \cdot \sin \overline{\frac{2\pi}{\lambda}(vt-x)} \cdot \sin\beta \cdot \sin \overline{\beta+\alpha+\phi} + c \cdot \cos \overline{\frac{2\pi}{\lambda}(vt-x)} \cdot \cos\beta \cdot \cos \overline{\beta+\alpha+\phi}$$

and for the extraordinary ray

$$-c \cdot \sin \overline{\frac{2\pi}{\lambda}(vt-x)} \cdot \sin\beta \cdot \cos \overline{\beta+\alpha+\phi} + c \cdot \cos \overline{\frac{2\pi}{\lambda}(vt-x)} \cdot \cos\beta \cdot \sin \overline{\beta+\alpha+\phi}.$$

After emerging from the crystal, we must (as before) diminish x in the latter expression by Θ: and thus the vibration of the extraordinary ray

$$= -c.\sin\overline{\frac{2\pi}{\lambda}(vt-x+\Theta)}.\sin\beta.\cos\overline{\beta+\alpha+\phi}$$

$$+c.\cos\overline{\frac{2\pi}{\lambda}(vt-x+\Theta)}.\cos\beta.\sin\overline{\beta+\alpha+\phi}.$$

The only parts of these transmitted by the analyzing plate are the resolved parts perpendicular to its plane of polarization = vibration of ordinary ray $\times\cos\phi$ + that of extraordinary ray $\times\sin\phi$

$$= c.\sin\overline{\frac{2\pi}{\lambda}(vt-x)}.\sin\beta.\sin\overline{\beta+\alpha+\phi}.\cos\phi$$

$$+c.\cos\overline{\frac{2\pi}{\lambda}(vt-x)}.\cos\beta.\cos\overline{\beta+\alpha+\phi}.\cos\phi$$

$$-c.\sin\overline{\frac{2\pi}{\lambda}(vt-x+\Theta)}.\sin\beta.\cos\overline{\beta+\alpha+\phi}.\sin\phi$$

$$+c.\cos\overline{\frac{2\pi}{\lambda}(vt-x+\Theta)}.\cos\beta.\sin\overline{\beta+\alpha+\phi}.\sin\phi.$$

The coefficient of $\sin\overline{\frac{2\pi}{\lambda}(vt-x)}$ is

$$c.\sin\beta.\sin\overline{\beta+\alpha+\phi}.\cos\phi-c.\cos\frac{2\pi}{\lambda}\Theta.\sin\beta.\cos\overline{\beta+\alpha+\phi}.\sin\phi$$

$$-c.\sin\frac{2\pi}{\lambda}\Theta.\cos\beta.\sin\overline{\beta+\alpha+\phi}.\sin\phi:$$

the coefficient of $\cos\overline{\frac{2\pi}{\lambda}(vt-x)}$ is

$$c.\cos\beta.\cos\overline{\beta+\alpha+\phi}.\cos\phi-c.\sin\frac{2\pi}{\lambda}\Theta.\sin\beta.\cos\overline{\beta+\alpha+\phi}.\sin\phi$$

$$+c.\cos\frac{2\pi}{\lambda}\Theta.\cos\beta.\sin\overline{\beta+\alpha+\phi}.\sin\phi.$$

The intensity, or the sum of the squares of these coefficients, is (after reduction)

$$\frac{c^2}{2}\{1+\cos 2\beta.\cos 2\phi.\cos 2.\overline{\beta+\alpha+\phi}$$

$$+\cos\frac{2\pi}{\lambda}\Theta.\cos 2\beta.\sin 2\phi.\sin 2.\overline{\beta+\alpha+\phi}-\sin\frac{2\pi}{\lambda}\Theta.\sin 2\beta.\sin 2\phi\}.$$

Now if $\beta=45^0$, $\cos 2\beta=0$; $\sin 2\beta=1$;

$$\text{and the intensity } =\frac{c^2}{2}\left\{1-\sin\frac{2\pi}{\lambda}\Theta.\sin 2\phi\right\}.$$

1st. Since α has disappeared from this expression, the figure will be the same whatever be the value of α, that is, whatever be the position of the analyzing plate.

2nd. When $\phi=0$, $=90^0$, $=180^0$, $=270^0$, the expression becomes $\frac{c^2}{2}$ which shews that there is a faint cross parallel and perpendicular to the plane of reflexion at the analyzing plate.

3rd. When ϕ is $>0<90^0$, or $>180^0<270^0$, the intensity is greatest if $\frac{2\pi}{\lambda}\Theta=\frac{3\pi}{2}$, $\frac{7\pi}{2}$, &c. and least if $\frac{2\pi}{\lambda}\Theta=\frac{\pi}{2}$, $\frac{5\pi}{2}$, &c. When ϕ is $>90^0<180^0$, or $>270^0<360^0$, the intensity is least if $\frac{2\pi}{\lambda}\Theta=\frac{3\pi}{2}$, $\frac{7\pi}{2}$, &c. and greatest if $\frac{2\pi}{\lambda}\Theta=\frac{\pi}{2}$, $\frac{5\pi}{2}$, &c.

4th. If $\beta=135^0$, the expression becomes

$$\frac{c^2}{2}\left\{1+\sin\frac{2\pi}{\lambda}\Theta.\sin 2\phi\right\}:$$

from which it is easily seen that the bright parts of the quadrantal rings in this case correspond to the faint ones when $\beta=45^0$: and vice versâ.

III. If in the last experiment the rhomb be placed in position 0, we must make $\beta=0$, which gives for the intensity

$$\frac{c^2}{2}\left\{1+\cos 2\phi.\cos 2.\overline{\alpha+\phi}+\cos\frac{2\pi}{\lambda}\Theta.\sin 2\phi.\sin 2.\overline{\alpha+\phi}\right\},$$

exactly as in the general case of experiment I.

If the rhomb be in position 90°, the expression is exactly the same.

If the analyzing and polarising plates be crossed, we must make $\alpha = 90^\circ$, and the general expression becomes

$$\frac{c^2}{2}\{1 - \cos 2\beta \cdot \cos 2\phi \cdot \cos 2 \cdot \overline{\beta+\phi}$$

$$- \cos\frac{2\pi}{\lambda}\Theta \cdot \cos 2\beta \cdot \sin 2\phi \cdot \sin 2 \cdot \overline{\beta+\phi} - \sin\frac{2\pi}{\lambda}\Theta \cdot \sin 2\beta \cdot \sin 2\phi\}.$$

Here if $\phi = 0$, $= 90^\circ$, $= 180^\circ$, $= 270^\circ$, the intensity is $\frac{c^2}{2}\sin^2 2\beta$, which shews that there is a faint cross parallel and perpendicular to the plane of reflection. For other values of ϕ the only variable part is comprehended in the two last terms or

$$-\frac{c^2}{2}\sin 2\phi \cdot \{\sin 2\beta \cdot \sin\frac{2\pi}{\lambda}\Theta + \cos 2\beta \cdot \sin 2 \cdot \overline{\beta+\phi} \cdot \cos\frac{2\pi}{\lambda}\Theta\}.$$

This may be put under the form $A\cos\left(\frac{2\pi}{\lambda}\Theta - B\right)$,

where

$$A = -\frac{c^2}{2}\sin 2\phi \sqrt{\sin^2 2\beta + \cos^2 2\beta \cdot \sin^2 2 \cdot \overline{\beta+\phi}}, \text{ and } \tan B = \frac{\tan 2\beta}{\sin 2 \cdot \overline{\beta+\phi}}.$$

The equation to the dark rings will be found by making

$$\frac{2\pi}{\lambda}\Theta - B = 0, \text{ or } = 2\pi, \text{ or } = 4\pi, \text{ \&c.};$$

$$\text{hence } \Theta = \frac{\lambda B}{2\pi}, \text{ or } = \lambda + \frac{\lambda B}{2\pi}, \text{ or } = 2\lambda + \frac{\lambda B}{2\pi}, \text{ \&c.}$$

$$\text{and } \theta = \sqrt{\frac{2b\lambda}{T(a^2-b^2)}} \cdot \sqrt{\frac{B}{2\pi}}, \text{ or } = \sqrt{\frac{2b\lambda}{T(a^2-b^2)}} \cdot \sqrt{1+\frac{B}{2\pi}},$$

$$\text{or } = \sqrt{\frac{2b\lambda}{T(a^2-b^2)}}\sqrt{2+\frac{B}{2\pi}}, \text{ \&c.}$$

Now when β is small, $\sin 2 \cdot \overline{\beta+\phi}$ being positive, $\tan 2\beta$ is small, and $\tan B$ is small: except $\sin 2 \cdot \overline{\beta+\phi}$ is small, when B sud-

denly becomes $=90°$: when $\sin 2.\overline{\beta+\phi}$ changes its sign, B changes its sign, and is $=-90°$; its magnitude then diminishes till $\sin 2.\overline{\beta+\phi}=-1$: and it then goes through the same changes. The circle is therefore changed into the form represented in fig. 17. But the rings at the parts where $\sin 2.\overline{\beta+\phi}=0$ being very faint, the bends of the curve scarcely attract the attention, and the figure appears elliptical. But when β increases, the intensity of the rings where $\sin 2.\overline{\beta+\phi}=0$ is not small, (it is represented by A), and the change of form is easily seen*. All these conclusions correspond perfectly with observation.

IV. To investigate in a similar manner the appearances presented by plates of quartz, on the suppositions made in the beginning of this paper, we must resolve a plane-polarized† ray into two elliptically polarized rays. In fig. 18. let AP_1 be the plane of primitive polarization: AC the principal plane of the crystal; the ordinary ray being elliptically polarized, will consist of one vibration in the direction Oo_1, and another in the direction $o_1 o_2$ following it one quarter of an interval of undulations: the coefficient of the latter vibration being $=k\times$ that of the former, where k is a fraction depending by some unknown law on the inclination to the axis, but becoming $=1$ when the inclination $=0$, and $=0$ when the inclination is considerable. And the extraordinary ray will consist of one vibration in the direction Ee_1, and another in e_1e_2 preceding it one quarter of an interval: the coefficient of

* I have investigated (in nearly the same manner) the form of the curves, supposing the crystal placed between the polarizing plate and the rhomb. The calculated phenomena are nearly the same as those described, and agree perfectly with observations.

† I use this term instead of *rectilinearly-polarized*, the natural derivative from Fresnel's substantive, only because it is shorter.

the latter being $= \frac{1}{k} \times$ that of the former. Let the vibration perpendicular to AP_1 be $c.\sin\frac{2\pi}{\lambda}(vt-x)$ or $c.\sin\xi$ (for brevity): and let the vibrations be

In Oo_1, $p.\sin\xi+v$, or $p\cos v.\sin\xi+p\sin v.\cos\xi$, or $w\sin\xi+x\cos\xi$.

In o_1o_2, $kp.\sin\xi+v-90°$,

or $-kp.\cos\xi+v$, or $kp\sin v.\sin\xi-kp\cos v.\cos\xi$, or $kx\sin\xi-kw\cos\xi$.

In Ee_1, $q.\sin\xi+\chi$, or $q\cos\chi.\sin\xi+q.\sin\chi.\cos\xi$, or $y.\sin\xi+z.\cos\xi$.

In e_1e_2, $\frac{q}{k}.\sin\xi+\chi+90°$,

or $\frac{q}{k}.\cos\xi+\chi$ or $-\frac{q}{k}\sin\chi.\sin\xi+\frac{q}{k}\cos\chi.\cos\xi$, or $-\frac{z}{k}\sin\xi+\frac{y}{k}\cos\xi$.

Resolving these in directions parallel and perpendicular to AP_1, and comparing them with the vibration from the original polarization,

$$\cos\alpha+\phi\,(w\sin\xi+x\cos\xi)+\sin\alpha+\phi\,(kx\sin\xi-kw\cos\xi)$$

$$+\cos\alpha+\phi\,(y\sin\xi+z\cos\xi)+\sin\alpha+\phi\,(-\tfrac{z}{k}\sin\xi+\tfrac{y}{k}\cos\xi)=c\sin\xi,$$

$$\sin\alpha+\phi\,(w\sin\xi+x\cos\xi)-\cos\alpha+\phi\,(kx\sin\xi-kw\cos\xi)$$

$$+\sin\alpha+\phi\,(y\sin\xi+z\cos\xi)-\cos\alpha+\phi\,(-\tfrac{z}{k}\sin\xi+\tfrac{y}{k}\cos\xi)=0.$$

Or (since these equations ought to hold for all values of ξ) equating separately the coefficients of $\sin\xi$ and $\cos\xi$,

$$\cos\alpha+\phi.w+k.\sin\alpha+\phi.x+\cos\alpha+\phi.y-\frac{\sin\alpha+\phi}{k}z=c \quad \ldots\ldots (1),$$

$$\cos\alpha+\phi.x-k\sin\alpha+\phi.w+\cos\alpha+\phi.z+\frac{\sin\alpha+\phi}{k}y=0 \quad \ldots\ldots (2),$$

$$\sin\overline{\alpha+\phi}\,.\,w-k\,.\cos\overline{\alpha+\phi}\,.\,x+\sin\overline{\alpha+\phi}\,.\,y+\frac{\cos\overline{\alpha+\phi}}{k}z=0 \ \ldots\ldots\ (3),$$

$$\sin\overline{\alpha+\phi}\,.\,x+k\,.\cos\overline{\alpha+\phi}\,.\,w+\sin\overline{\alpha+\phi}\,.\,z-\frac{\cos\overline{\alpha+\phi}}{k}y=0 \ \ldots\ldots\ (4).$$

By the solution of these equations

$$w=\frac{c}{1+k^2}\cos\overline{\alpha+\phi},$$

$$x=\frac{k}{1+k^2}c\,.\sin\overline{\alpha+\phi},$$

$$y=\frac{k^2}{1+k^2}c\,.\cos\overline{\alpha+\phi},$$

$$z=-\frac{k}{1+k^2}c\,.\sin\overline{\alpha+\phi}.$$

And hence the expressions for the vibrations are (omitting the common factor $\frac{c}{1+k^2}$):

In Oo_1, $\quad\cos\overline{\alpha+\phi}\,.\sin\overline{\frac{2\pi}{\lambda}(vt-x)}+k\,.\sin\overline{\alpha+\phi}\,.\cos\overline{\frac{2\pi}{\lambda}(vt-x)}.$

In o_1o_2, $\quad k^2\,.\sin\overline{\alpha+\phi}\,.\sin\overline{\frac{2\pi}{\lambda}(vt-x)}-k\,.\cos\overline{\alpha+\phi}\,.\cos\overline{\frac{2\pi}{\lambda}(vt-x)}.$

In Ee_1, $\quad k^2\,.\cos\overline{\alpha+\phi}\,.\sin\overline{\frac{2\pi}{\lambda}(vt-x)}-k\,.\sin\overline{\alpha+\phi}\,.\cos\overline{\frac{2\pi}{\lambda}(vt-x)}.$

In e_1e_2, $\quad\sin\overline{\alpha+\phi}\,.\sin\overline{\frac{2\pi}{\lambda}(vt-x)}+k\,.\cos\overline{\alpha+\phi}\,.\cos\overline{\frac{2\pi}{\lambda}(vt-x)}.$

After emerging from the crystal, x must be diminished, in the two latter expressions, by Θ. (Θ is in fact negative for quartz, but that circumstance makes no difference in our investigations or conclusions).

Now when the light is received on the analyzing plate, the only parts sensible are those perpendicular to its plane of reflexion. They are, (putting ξ as before for $\frac{2\pi}{\lambda}(vt-x)$),

From Oo_1, $\cos\overline{\alpha+\phi}.\cos\phi.\sin\xi+k.\sin\overline{\alpha+\phi}.\cos\phi.\cos\xi.$

From o_1o_2, $k^2.\sin\overline{\alpha+\phi}.\sin\phi.\sin\xi-k.\cos\overline{\alpha+\phi}.\sin\phi.\cos\xi.$

From Ee_1, $k^2.\cos\overline{\alpha+\phi}.\cos\phi.\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}}-k.\sin\overline{\alpha+\phi}.\cos\phi.\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}}.$

From e_1e_2, $\sin\overline{\alpha+\phi}.\sin\phi.\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}}+k.\cos\overline{\alpha+\phi}.\sin\phi.\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}}.$

Taking the sum, the coefficient of $\sin\xi$ is

$$\cos\overline{\alpha+\phi}.\cos\phi+k^2.\sin\overline{\alpha+\phi}.\sin\phi+k^2.\cos\overline{\alpha+\phi}.\cos\phi.\cos\frac{2\pi\Theta}{\lambda}$$
$$+k.\sin\alpha.\sin\frac{2\pi\Theta}{\lambda}+\sin\overline{\alpha+\phi}.\sin\phi.\cos\frac{2\pi\Theta}{\lambda}.$$

The coefficient of $\cos\xi$ is

$$k\sin\alpha+k^2.\cos\overline{\alpha+\phi}.\cos\phi.\sin\frac{2\pi\Theta}{\lambda}-k\sin\alpha.\cos\frac{2\pi\Theta}{\lambda}+\sin\overline{\alpha+\phi}.\sin\phi.\sin\frac{2\pi\Theta}{\lambda}.$$

The sum of the squares of the coefficients is (after all reductions)

$$(1-k^4)^2.\cos^2\overline{\alpha+2\phi}.\sin^2\frac{\pi\Theta}{\lambda}+\left(\overline{1+k^4}.\cos\alpha.\cos\frac{\pi\Theta}{\lambda}+2k.\sin\alpha.\sin\frac{\pi\Theta}{\lambda}\right)^2.$$

And restoring the multiplier $\frac{c^2}{(1+k^2)^2}$, we have for the brightness

$$c^2\left(\frac{1-k^2}{1+k^2}\right)^2.\cos^2.\overline{\alpha+2\phi}.\sin^2\frac{\pi\Theta}{\lambda}+c^2\left(\cos\alpha.\cos\frac{\pi\Theta}{\lambda}+\frac{2k}{1+k^2}\sin\alpha.\sin\frac{\pi\Theta}{\lambda}\right)^2.$$

When the plates are crossed, or $\alpha = 90^0$, this becomes

$$c^2\left(\frac{1-k^2}{1+k^2}\right)^2 . \sin^2 2\phi . \sin^2\frac{\pi\Theta}{\lambda} + c^2 . \frac{4k^2}{(1+k^2)^2} . \sin^2\frac{\pi\Theta}{\lambda}$$

$$= c^2 . \sin^2\frac{\pi\Theta}{\lambda}\left\{\frac{4k^2}{(1+k^2)} + \overline{\frac{1-k^2}{1+k^2}\Big|^2} \sin^2 2\phi\right\}.$$

1. For any value of ϕ this is 0 when

$$\frac{\pi\Theta}{\lambda} = 0, \quad = \pi, \quad = 2\pi, \quad \&c.$$

This shews that there are dark rings, exactly circular: it represents correctly the experimental fact.

2. But since by our 4th hypothesis the spheroid and the sphere, used for determining the course of the two rays, do not touch, Θ will have some value when θ is 0. It cannot therefore be expressed simply by

$$T \times \frac{a^2 - b^2}{2b} \times \theta^2,$$

but must have an additional term $T \times E$. The value of E (as depending on λ) may be thus found. At the center k is supposed (hypothesis 3) to $= 1$. Consequently the general expression for the intensity of light at the center is

$$c^2\left(\cos\alpha . \cos\frac{\pi\Theta}{\lambda} + \sin\alpha \sin\frac{\pi\Theta}{\lambda}\right)^2 \text{ or } c^2 . \cos^2\left(\alpha - \frac{\pi\Theta}{\lambda}\right).$$

This is 0 when $\alpha - \frac{\pi\Theta}{\lambda} = 90^0$: or putting Θ', α', for these particular values of Θ and α, $\alpha' = 90 + \frac{\pi\Theta'}{\lambda}$. Now it was found by M. Biot that in a right-handed crystal, α' (measured in the direction that we have supposed) must exceed 90^0 by a quantity proportional to the thickness of the plate directly and the square of λ inversely. That is,

$$\frac{\pi\Theta'}{\lambda} = \frac{eT}{\lambda^2}; \text{ or } \Theta' = \frac{eT}{\lambda\pi};$$

E therefore is $\frac{e}{\lambda\pi}$, and consequently $\Theta = T \times \left(\frac{e}{\lambda\pi} + \frac{a^2 - b^2}{2b}\theta^2\right)$.

We may remind the reader (as we shall have occasion to use it afterwards) that $\frac{eT}{\lambda^2}$ is the angle through which the analyzing plate must be turned from the crossed position to produce darkness at the center for the particular colour used. We may also remark that, if we still consider Θ as positive (which we shall continue to do), all our expressions (as appears from this comparison of theory and observation) must be understood to apply to a right-handed crystal: if the sign of k be changed, they will apply to a left-handed crystal. But if, more correctly, we put a negative symbol for Θ, then our expressions would apply to a left-handed crystal, unless the sign of k were changed.*

The value of $\frac{\pi\Theta}{\lambda}$ is therefore $\frac{Te}{\lambda^2} + \frac{T\pi(a^2 - b^2)}{2b\lambda}\theta^2$. Let us suppose that $\lambda' + \delta\lambda$ may represent the length for rays of any colour, λ' being that for one of the mean rays, and therefore constant, and $\delta\lambda$ being small for all the bright colours. Then for $\frac{\pi\Theta}{\lambda' + \delta\lambda}$ we have

$$\frac{Te}{\lambda'^2} + \frac{T\pi(a^2 - b^2)}{2b\lambda'}\theta^2 - \frac{\delta\lambda}{\lambda'^2}\left(\frac{2Te}{\lambda'} + \frac{T\pi(a^2 - b^2)}{2b}\theta^2\right) \text{ nearly.}$$

The mixture of colours in the rings will depend only on the difference of the values of $\frac{\pi\Theta}{\lambda}$ for different colours, and not on its absolute values, and

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* The reader who will take the trouble of tracing the expressions will find that, if the sign of Θ and of k be changed at the same time, not only will the right-handed-ness of the crystal remain the same, on comparing the expression with Biot's experiment, but also all the directions of the spirals &c. in the succeeding experiments will remain the same. Thus the connection between the right-handed-ness and the direction of the spirals is independent of the sign assumed for Θ. With this consideration, I have thought it best to use the same symbol in the theorems for calc spar and for quartz.

consequently only on the last term of this expression.* Now this is the same that we should have had for the colours in Newton's scale depending on the thickness $\frac{2Te}{\lambda'} + \frac{T\pi(a^2-b^2)}{2b}\theta^2$ of a plate of air, if $\frac{a^2-b^2}{2b}$ be the same for all colours: or if the variations of $\frac{a^2-b^2}{2b}$ be proportional to the variations of λ, the last term must be altered in a certain proportion, and our statement will still be true. It appears therefore that the central tint will be nearly that corresponding in Newton's scale to a plate of air whose thickness is $\frac{2Te}{\lambda'}$: and will be followed by the other orders of Newton's scale. This agrees sufficiently with the observations.

3. Unless $\frac{\pi\Theta}{\lambda} = 0$, or $= \pi$, &c. the light is not $= 0$: therefore there is no black cross. But the light is least when $\phi = 0$, or $= 90^0$, or $= 180^0$, or $= 270^0$: and greatest when $\phi = 45^0$, or $= 135^0$, or $= 225^0$, or $= 315^0$. This shews that there are dark brushes parallel and perpendicular to the plane of reflexion, but not interrupting the rings. As k is nearly $= 1$ in the neighbourhood of the axis (by hypothesis) they are not sensible near the center: but as, on removing to a distance, k approaches to 1, they become stronger. This is conformable to observation.

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* This is not strictly true: for though, in comparing this with one of Newton's rings, the colours mixed may be the same, their intensities will not be equal except $\sin^2\frac{\pi\Theta}{\lambda}$ be the same, and consequently the compound colour will not be the same. But it is plain that, if we take the ring preceding and that followiug the point where the difference of $\frac{\pi\Theta}{\lambda}$ is the same as in the case before us, then at the points where $\sin^2\frac{\pi\Theta}{\lambda}$ has the same value for the mean rays, we shall have mixtures differing in opposite ways from that under consideration. In the same manner, in the rest of the paragraph, it must be understood that, if we take those colours of Newton's scale between which the colours of these rings lie, we shall always advance in the scale: but we may possibly have no ring intermediate between two of Newton's, or we may possibly have more than one.

V. Taking the general expression for the brightness, and making $\tan\psi = \frac{2k}{1+k^2}\tan\alpha$, we have

$$\cos\alpha\,.\cos\frac{\pi\Theta}{\lambda} + \frac{2k}{1+k^2}\sin\alpha\,.\sin\frac{\pi\Theta}{\lambda} = \sqrt{\cos^2\alpha + \frac{4k^2}{(1+k^2)^2}\sin^2\alpha}\,.\cos\overline{\frac{\pi\Theta}{\lambda} - \psi}$$

and the general expression for the brightness is

$$c^2\left(\frac{1-k^2}{1+k^2}\right)^2.\cos^2\overline{\alpha+2\phi}\,.\sin^2\frac{\pi\Theta}{\lambda} + c^2\left(\cos^2\alpha + \frac{4k^2}{(1+k^2)^2}\sin^2\alpha\right).\cos^2\overline{\frac{\pi\Theta}{\lambda} - \psi}.$$

Supposing that k is not much altered by a small alteration of Θ, this is a maximum or minimum for a given value of ϕ if

$$0 = (1-k^2)^2.\cos^2\overline{\alpha+2\phi}\,.\sin\frac{2\pi\Theta}{\lambda} - \left(\overline{1+k^2}\right|^2.\cos^2\alpha + 4k^2\sin^2\alpha).\sin\overline{\frac{2\pi\Theta}{\lambda} - 2\psi},$$

or $\tan\overline{\frac{2\pi\Theta}{\lambda} - \psi}$

$$= \frac{2k}{1+k^2}\tan\alpha \times \frac{\left(\overline{1+k^2}\right|^2.\cos^2\alpha + 4k^2\sin^2\alpha + \left(\overline{1-k^2}\right|^2.\cos^2\overline{\alpha+2\phi}}{\left(\overline{1+k^2}\right|^2.\cos^2\alpha + 4k^2\sin^2\alpha - \left(\overline{1-k^2}\right|^2.\cos^2\overline{\alpha+2\phi}}.$$

Therefore $\frac{2\pi\Theta}{\lambda}$ will be greater than ψ (or $\psi+\pi$, or $\psi+2\pi$, or $\psi+3\pi$, &c. for each of these satisfies the equation $\tan\psi = \frac{2k}{1+k^2}\tan\alpha$) by the angle ω whose tangent is the second side of the expression.

1. If k be nearly equal to 1 (which we suppose to be true when θ is small) and α less than 90^0, the expression for $\tan\omega$ is always positive: its greatest value, when

$$\phi = 90^0 - \frac{\alpha}{2},\ \text{or}\ 180^0 - \frac{\alpha}{2},\ \text{or}\ 270^0 - \frac{\alpha}{2},\ \text{or}\ 360^0 - \frac{\alpha}{2},$$

$$\text{is}\ \frac{2k}{1+k^2}\tan\alpha \times \frac{2+2k^4-\left(\overline{1-k^2}\right|^2\sin^2\alpha}{4k^2-\left(\overline{1-k^2}\right|^2.\sin^2\alpha}:$$

and its least, when $\phi = 45^0 - \frac{\alpha}{2}$, or $135^0 - \frac{\alpha}{2}$, or $225^0 - \frac{\alpha}{2}$, or $315^0 - \frac{\alpha}{2}$, is $\frac{2k}{1+k^2}\tan\alpha$.

Therefore in the bright or dark rings $\frac{2\pi\Theta}{\lambda}$ will be greater than ψ, or $\psi + \pi$, or $\psi + 2\pi$, &c. by an angle ω which is always included between 0 and 90°, and which has its maximum at a part which, when α is less than 90°, is found by looking a little to the negative side of the perpendicular and parallel to the plane of reflexion (thus, if the crystal be right handed, we must look to the left of the upper part, and the points of the square appearance will be found in that place). When α is greater than 90°, the maximum value of the tangent takes place for points found by looking a little to the positive side of the perpendicular and parallel: but the tangent is then negative (for tan α which enters as a multiplier is negative). Consequently the maximum contraction of the circle is found by looking to the positive side, and the points of the squares will be found by looking to the negative side. Whichever therefore be the direction in which the analyzing plate is turned, the circles will be changed into the form represented in fig. 15 (the crystal being supposed right handed). This remarkable conclusion agrees perfectly with the facts of observation.

2. If k is very small, the expression for tan ω may become negative, which shews that ω will suddenly exceed 90°, and after having continued so during the change of ϕ through an arc of various extent (according to the value of α) will suddenly become less than 90°. This shews that the form of the bright and dark rings will be that of fig. 2, except that instead of absolute interruption of the rings by the eight radii, they will pass very highly inclined through those radii. The rings of quartz become so faint at a distance from the center that I have not been able to observe whether this is or is not supported by fact.

3. We have already noticed that when $\alpha = 90° + \frac{eT}{\lambda^2}$ there is a dark spot at the center. Now for any given value of Θ, it appears (from the

general expression) that the light is least when

$$\phi = 45^0 - \frac{\alpha}{2},\ 135^0 - \frac{\alpha}{2},\ \&c.$$

This shews that, with this value of α, the spot will be a darkish cross, its arms in positions

$$45 - \frac{\alpha}{2},\ 135^0 - \frac{\alpha}{2},\ \&c.$$

But these are exactly the angles at which the depression of the ring below a circle is a minimum, or at which the points of the square are found. Therefore, we may expect to find a cross-like spot in the center, its arms in the diagonals of the square. This corresponds perfectly with the phœnomena.

4. The succession of colours in the cross-like spot, it is easily seen, depends only on this circumstance: that as λ is greater for red than for blue rays, the value of α which allows no red rays to pass is less than that which allows no blue rays to pass. That is, for a certain small value of α, there are rays of the blue and only of the spectrum transmitted: for a larger value, the rays only of the red end are transmitted.

5. If the polarizing and analyzing plates are parallel, or $\alpha = 0$, the expression for the brightness becomes

$$c^2 - c^2\{1 - \left|\frac{1-k^2}{1+k^2}\right|^2 . \cos^2 2\phi\} \sin^2 \frac{\pi\Theta}{\lambda}.$$

It is easily seen that this indicates a series of rings, but there is now no total darkness. When

$$\phi = 0,\ = 90^0,\ = 180^0,\ = 270^0,$$

the expression is

$$c^2 - c^2 . \frac{4k^2}{(1+k^2)^2} \sin^2 \frac{\pi\Theta}{\lambda}.$$

As k becomes smaller, this varies less with the variations of θ, but does not vanish: that is, in receding from the center, the rings are more and more interrupted by a white cross. If the plate of quartz be thin, it may happen that the first ring is so large as to be sensibly interrupted. In this case (as the first ring is broad) it will lose the appearance of four interrupted quadrants and become four dots. This is easily seen in experiment.

VI. We have seen that, for a bright or dark ring, $\frac{2\pi\Theta}{\lambda}$ will differ from ψ only by ω. When α is 0, ω is 0: and when α is 90^0, ω is 90^0, having increased gradually to that value. Also when α is 0, ψ is 0 or π, or 2π, &c.: and as α increases gradually to 90^0, ψ increases gradually to

$$\frac{\pi}{2}, \text{ or } \frac{3\pi}{2}, \text{ or } \frac{5\pi}{2}, \text{ &c.}$$

Consequently, $\frac{2\pi\Theta}{\lambda}$ or $\psi + \omega$ increases by 180^0, while α increases from 0 to 90^0. In the same manner, as α increases gradually from 90^0 to 180^0, ω increases gradually from 90^0 to 180^0, and ψ from

$$\frac{\pi}{2}, \text{ or } \frac{3\pi}{2}, \text{ or } \frac{5\pi}{2}, \text{ &c. to } \pi, \text{ or } 2\pi, \text{ or } 3\pi, \text{ &c.:}$$

and consequently $\frac{2\pi\Theta}{\lambda}$ or $\psi + \omega$ again increases gradually by 180^0. The same holds for every successive quadrant of revolution. Thus, if we fix our attention on any ring, and turn the analyzing plate to the left (the crystal being right-handed) the ring's distance from the center (or θ, which

$$= \sqrt{\frac{2b}{T(a^2 - b^2)}\left(\Theta - \frac{Te}{\pi\lambda}\right)}\Big)$$

will increase continually, but not uniformly. This is conformable to fact.

VII. When Fresnel's rhomb is interposed in position 45°, we must suppose AR, fig. 18, to be perpendicular to the plane of internal reflection, and we shall have the vibrations

$-\frac{c}{\sqrt{2}}\sin\frac{2\pi}{\lambda}(vt-x)$ parallel to AR, and $\frac{c}{\sqrt{2}}\cos\frac{2\pi}{\lambda}(vt-x)$ perpendicular to AR. Resolving these in directions parallel and perpendicular to AC, we find for the former

$$-\frac{c}{\sqrt{2}}\cdot\sin\frac{2\pi}{\lambda}(vt-x)\cdot\cos(\alpha+\phi+45^\circ)+\frac{c}{\sqrt{2}}\cdot\cos\frac{2\pi}{\lambda}(vt-x)\cdot\sin(\alpha+\phi+45^\circ),$$

$$\text{or } -\frac{c}{\sqrt{2}}\sin\frac{2\pi}{\lambda}(vt-x)-(\alpha+\phi)-45^\circ:$$

and for the latter,

$$\frac{c}{\sqrt{2}}\sin\frac{2\pi}{\lambda}(vt-x)\cdot\sin(\alpha+\phi+45^\circ)+\frac{c}{\sqrt{2}}\cos\frac{2\pi}{\lambda}(vt-x)\cdot\cos(\alpha+\phi+45^\circ),$$

$$\text{or } \frac{c}{\sqrt{2}}\cos\frac{2\pi}{\lambda}(vt-x)-(\alpha+\phi)-45^\circ.$$

Now taking the same expressions as before, for the vibrations in Oo_1, &c. and comparing the sums in directions parallel and perpendicular to AC, (putting $\frac{2\pi}{\lambda}(vt-x)-45^\circ=\xi$), we have

$$-\frac{c}{\sqrt{2}}\sin(\xi-(\alpha+\phi)) \text{ or } -\frac{c}{\sqrt{2}}\cos(\alpha+\phi)\cdot\sin\xi+\frac{c}{\sqrt{2}}\sin(\alpha+\phi)\cdot\cos\xi$$

$$=\left(kx-\frac{z}{k}\right)\sin\xi+\left(\frac{y}{k}-kw\right)\cos\xi$$

$$\frac{c}{\sqrt{2}}\cos(\xi-(\alpha+\phi)) \text{ or } \frac{c}{\sqrt{2}}\sin(\alpha+\phi)\cdot\sin\xi+\frac{c}{\sqrt{2}}\cos(\alpha+\phi)\cdot\cos\xi$$

$$=(w+y)\sin\xi+(x+z)\cos\xi$$

Comparing the coefficients of sin ξ and of cos ξ,

$$kx - \frac{z}{k} = -\frac{c}{\sqrt{2}} \cos \overline{\alpha + \phi}$$

$$\frac{y}{k} - kw = \frac{c}{\sqrt{2}} \sin \overline{\alpha + \phi}$$

$$w + y = \frac{c}{\sqrt{2}} \sin \overline{\alpha + \phi}$$

$$x + z = \frac{c}{\sqrt{2}} \cos \overline{\alpha + \phi}.$$

Obtaining the values of w, x, y, z, and substituting in the expressions for the vibrations, we find (omitting the common factor $\frac{c}{(1+k^2)\sqrt{2}}$).

In Oo_1 $\quad (1-k) . \cos \overline{\xi - (\alpha+\phi)}.$

In o_1o_2 $\quad k(1-k) . \sin \overline{\xi - (\alpha+\phi)}.$

In Ee_1 $\quad k(1+k) . \cos \overline{\xi - (\alpha+\phi)}.$

In e_1e_2 $\quad -(1+k) . \sin \overline{\xi - (\alpha+\phi)}.$

After emerging from the crystal, ξ in the two latter expressions must be increased by $\frac{2\pi\Theta}{\lambda}$.

Taking the resolved parts of the vibration perpendicular to AP_2, we find for the efficient vibration,

$$(1-k) . \cos \overline{\xi - (\alpha+\phi)} . \cos \phi + k(1-k) . \sin \overline{\xi - (\alpha+\phi)} . \sin \phi$$

$$+ k(1+k) . \cos \overline{\xi - (\alpha+\phi) + \frac{2\pi\Theta}{\lambda}} . \cos \phi - (1+k) . \sin \overline{\xi - (\alpha+\phi) + \frac{2\pi\Theta}{\lambda}} . \sin \phi.$$

The coefficient of $\sin \overline{\xi - (\alpha+\phi)}$ is

$$k(1-k) . \sin \phi - k(1+k) . \cos \phi . \sin \frac{2\pi\Theta}{\lambda} - (1+k) . \sin \phi . \cos \frac{2\pi\Theta}{\lambda}.$$

The coefficient of $\cos\overline{\xi-(a+\phi)}$ is

$$(1-k).\cos\phi + k\,(1+k).\cos\phi.\cos\frac{2\pi\Theta}{\lambda} - (1+k).\sin\phi.\sin\frac{2\pi\Theta}{\lambda}.$$

The sum of the squares is

$$\overline{1+k^2}|^2 - 2k.\overline{1-k^2}.\cos 2\phi + 2k.\overline{1-k^2}.\cos 2\phi.\cos\frac{2\pi\Theta}{\lambda}$$

$$-(1-k^4).\sin 2\phi.\sin\frac{2\pi\Theta}{\lambda}.$$

Restoring the factor $\frac{c^2}{2\,(1+k^2)^2}$, we have for the brightness

$$\frac{c^2}{2}\left\{1 - \frac{2k.(1-k^2)}{(1+k^2)^2}\cos 2\phi + \frac{2k\,(1-k^2)}{(1+k^2)^2}\cos 2\phi.\cos\frac{2\pi\Theta}{\lambda}\right.$$

$$\left. - \frac{1-k^2}{1+k^2}\sin 2\phi.\sin\frac{2\pi\Theta}{\lambda}\right\}.$$

If we make $\tan\chi = \frac{1+k^2}{2k}.\tan 2\phi$, the two last terms, or

$$\frac{1-k^2}{1+k^2}\left\{\frac{2k}{1+k^2}\cos 2\phi.\cos\frac{2\pi\Theta}{\lambda} - \sin 2\phi.\sin\frac{2\pi\Theta}{\lambda}\right\}$$

become $\frac{1-k^2}{1+k^2}\sqrt{\frac{4k^2}{(1+k^2)^2}\cos^2 2\phi + \sin^2 2\phi}.\cos\overline{\frac{2\pi\Theta}{\lambda}+\chi}$,

and the expression for the brightness is

$$\frac{c^2}{2}\left\{1 - \frac{1-k^2}{1+k^2}\left(\sqrt{\frac{4k^2}{(1+k^2)^2}\cos^2 2\phi + \sin^2 2\phi} + \frac{2k}{1+k^2}\cos 2\phi\right)\right.$$

$$\left. + 2\frac{1-k^2}{1+k^2}\sqrt{\frac{4k^2}{(1+k^2)^2}\cos^2 2\phi + \sin^2 2\phi}.\cos^2\overline{\frac{\pi\Theta}{\lambda}+\frac{\chi}{2}}\right\}.$$

1st. As the multiplier of $\cos^2\overline{\frac{\pi\Theta}{\lambda}+\frac{\chi}{2}}$ is never $= 0$ while k has any value between 0 and 1, the rings are not interrupted in any part of their circumference.

2nd. This multiplier however is small when k is nearly $= 1$. It is also small when k is small, if $\sin 2\phi = 0$, that is, when $\phi = 0$, $= 90°$, $= 180°$, $= 270°$. Hence there will be no rings very near the center: and at a distance from the center, they will be faint in the lines parallel and perpendicular to the plane of reflexion.

3rd. The form of the dark rings will be determined by making $\overline{\cos^2 \frac{\pi\Theta}{\lambda} + \frac{\chi}{2}} = 0$: and that of the bright ones by making it $= 1$. The first of these suppositions gives

$$\frac{\pi\Theta}{\lambda} + \frac{\chi}{2} = \frac{\pi}{2}, \text{ or } = \frac{3\pi}{2}, \text{ \&c.: and } \frac{\pi\Theta}{\lambda} = \frac{\pi}{2} - \frac{\chi}{2}, \text{ \&c.}$$

Now χ increases from 0 to 90°, from 90° to 180°, &c., while 2ϕ increases from 0 to 90°, from 90° to 180°, &c.: consequently χ never differs much from 2ϕ: and therefore for the dark rings

$$\frac{\pi\Theta}{\lambda} = \frac{\pi}{2} - \phi, \text{ or } = \frac{3\pi}{2} - \phi, \text{ \&c. nearly.}$$

That is Θ increases continually as ϕ diminishes: and consequently θ increases continually as ϕ diminishes. This shews that the curve is a spiral, and that (reckoning from the central fold) it is turned in a negative direction; the eye when fixed on a part above the center must turn to the left to trace the curve as it recedes from the center, supposing the crystal right-handed. If the crystal be left-handed, the sign of k must be changed: this changes the sign of χ, and the spiral is turned in the opposite direction. This agrees perfectly with observation.

4th. If we take the radius vector in the direction opposite to that corresponding to any point of the spiral, that is, if we increase ϕ by 180°, we find for the new values of Θ in the dark rings

$$\frac{\pi}{2} - \overline{\phi + 180°}, \quad \frac{3\pi}{2} - \overline{\phi + 180°}, \quad \frac{5\pi}{2} - \overline{\phi + 180°}, \text{ \&c.,}$$

$$\text{or } -\frac{\pi}{2} - \phi, \quad \frac{\pi}{2} - \phi, \quad \frac{3\pi}{2} - \phi, \text{ \&c. nearly.}$$

This is exactly the same series as that for the original direction of the radius vector: and therefore the values of θ are exactly the same for any radius vector as for that opposite. But the curve (as we have seen above) is spiral. These two conditions require that the form of the dark line be two similar spirals mutually inwrapping each other, their positions differing by 180°. But no other alteration of ϕ will give the same values of Θ in the dark rings. Consequently the form of the dark rings is two spirals, and only two, turning in the same direction, and in opposite positions. This remarkable conclusion is supported by fact.

5th. When 2ϕ is between 0 and 90°, χ is greater than 2ϕ: when 2ϕ is between 90 and 180°, χ is less than 2ϕ: and so for successive quadrants. That is, when ϕ is between 0 and 45°, χ is too great, or Θ too small, for a spiral of uniform approach to the center: when ϕ is between 45° and 90°, Θ is too great. This shews that the spiral will have a square appearance, the right-hand angles being higher than the center. This is precisely the form really presented to the eye.

6th. The expression for the brightness of the center (where $k=1$) is simply $\frac{c^2}{2}$. As this is independent of λ, it shews that there is the same mixture of colours at the center as in the light which we use: and that therefore with common light the center is white.

I should only take up the reader's time unnecessarily by going through the investigation with the rhomb in position 135°.

VIII. We have found (in the investigation of IV.) that putting ξ for $\frac{2\pi}{\lambda}(vt-x)$, the expressions for the vibrations when light, at first plane-polarized, has passed through a plate of right-handed quartz, are $\left(\text{omitting the common multiplier } \frac{c}{1+k^2}\right)$:

In Oo_1, $\quad \cos\overline{\alpha+\phi}.\sin\xi+k.\sin\overline{\alpha+\phi}.\cos\xi.$

In o_1o_2, $\quad k^2\sin\overline{\alpha+\phi}.\sin\xi-k.\cos\overline{\alpha+\phi}.\cos\xi.$

In Ee_1, $\quad k^2\cos\overline{\alpha+\phi}.\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}}-k.\sin\overline{\alpha+\phi}.\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}}.$

In e_1e_2, $\quad \sin\overline{\alpha+\phi}.\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}}+k\cos\overline{\alpha+\phi}.\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}}.$

We have now to resolve these into the elliptical vibrations in a plate of left-handed quartz of the same thickness, with its principal plane in the same position. It will be observed, that the difference of right-handed and left-handed quartz consists only in the difference of the sign of k.

Taking first the vibrations in Oo_1 and o_1o_2, and using those letters to express the vibrations in those directions,

$$Oo_1 = \text{vibration in } O'o'_1 + \text{that in } E'e'_1$$

$$= w\sin\xi + x\cos\xi + y\sin\xi + z\cos\xi,$$

$$o_1o_2 = \text{vibration in } o'_1o'_2 + \text{that in } e'_1e'_2$$

$$= -kx\sin\xi + kw\cos\xi + \frac{z}{k}\sin\xi - \frac{y}{k}\cos\xi.$$

Comparing the coefficients of sin ξ and cos ξ,

$$\left.\begin{aligned} \cos\overline{\alpha+\phi} &= w+y,\\ k\sin\overline{\alpha+\phi} &= x+z,\\ k^2\sin\overline{\alpha+\phi} &= -kx+\frac{z}{k},\\ -k\cos\overline{\alpha+\phi} &= kw-\frac{y}{k}, \end{aligned}\right\} \text{ whence } \left\{\begin{aligned} w &= \frac{1-k^2}{1+k^2}\cos\overline{\alpha+\phi},\\ x &= \frac{k(1-k^2)}{1+k^2}\sin\overline{\alpha+\phi},\\ y &= \frac{2k^2}{1+k^2}\cos\overline{\alpha+\phi},\\ z &= \frac{2k^3}{1+k^2}\sin\overline{\alpha+\phi}. \end{aligned}\right.$$

Whence the vibrations, after emerging from the second plate, are (omitting the common multiplier $\frac{1}{1+k^2}$, in addition to the former):

In $O'_1o'_1$ $\quad \overline{1-k^2}.\cos\overline{\alpha+\phi}.\sin\xi + k(1-k^2).\sin\overline{\alpha+\phi}.\cos\xi.$

In $o'_1o'_2$ $\quad -k^2(1-k^2).\sin\overline{\alpha+\phi}.\sin\xi + k(1-k^2).\cos\overline{\alpha+\phi}.\cos\xi.$

In $E'e'_1$ $\quad 2k^2.\cos\overline{\alpha+\phi}.\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}} + 2k^3.\sin\overline{\alpha+\phi}.\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}}.$

In $e'_1e'_2$ $\quad 2k^2.\sin\overline{\alpha+\phi}.\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}} - 2k\cos\overline{\alpha+\phi}.\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}}$

Similarly for the vibrations in Ee_1 and e_1e_2:

$$Ee_1 = w\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}} + x\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}} + y\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}} + z\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}}.$$

$$e_1e_2 = -kx\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}} + kw\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}} + \frac{z}{k}\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}} - \frac{y}{k}\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}}.$$

Comparing the coefficients of $\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}}$ and $\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}}$,

$$\left.\begin{aligned} k^2\cos\overline{\alpha+\phi} &= w+y,\\ -k\sin\overline{\alpha+\phi} &= x+z,\\ \sin\overline{\alpha+\phi} &= -kx+\frac{z}{k},\\ k\cos\overline{\alpha+\phi} &= kw-\frac{y}{k}, \end{aligned}\right\} \text{ whence } \left\{\begin{aligned} w &= \frac{2k^2}{1+k^2}\cos\overline{\alpha+\phi},\\ x &= \frac{-2k}{1+k^2}\sin\overline{\alpha+\phi},\\ y &= \frac{-k^2(1-k^2)}{1+k^2}\cos\overline{\alpha+\phi},\\ z &= \frac{k(1-k^2)}{1+k^2}\sin\overline{\alpha+\phi}. \end{aligned}\right.$$

Whence the vibrations, after emerging from the second plate, are $\left(\text{omitting } \frac{1}{1+k^2}\right)$:

$$\text{In } O'o'_1 \qquad 2k^2 . \cos\overline{\alpha+\phi} . \sin\overline{\xi + \frac{2\pi\Theta}{\lambda}}$$

$$-2k . \sin\overline{\alpha+\phi} . \cos\overline{\xi + \frac{2\pi\Theta}{\lambda}} .$$

$$\text{In } o'_1 o'_2 \qquad 2k^2 . \sin\overline{\alpha+\phi} . \sin\overline{\xi + \frac{2\pi\Theta}{\lambda}}$$

$$+2k^2 \cos\overline{\alpha+\phi} . \cos\overline{\xi + \frac{2\pi\Theta}{\lambda}} .$$

$$\text{In } E'e'_1 \quad -k^2(1-k^2)\cos\overline{\alpha+\phi} . \sin\overline{\xi + \frac{4\pi\Theta}{\lambda}}$$

$$+k(1-k^2)\sin\overline{\alpha+\phi} . \cos\overline{\xi + \frac{4\pi\Theta}{\lambda}} .$$

$$\text{In } e'_1 e'_2 \qquad (1-k^2) . \sin\overline{\alpha+\phi} . \sin\overline{\xi + \frac{4\pi\Theta}{\lambda}}$$

$$+k(1-k^2)\cos\overline{\alpha+\phi} . \cos\overline{\xi + \frac{4\pi\Theta}{\lambda}} .$$

Adding together the vibrations in the same direction, we find $\left(\text{omitting the factor } \frac{c}{(1+k^2)^2}\right)$:

Vibration parallel to Oo_1

$$= \overline{1-k^2} . \cos\overline{\alpha+\phi} . \sin\xi + k . (1-k^2) . \sin\overline{\alpha+\phi} . \cos\xi$$

$$+ 4k^2 . \cos\overline{\alpha+\phi} . \sin\overline{\xi + \frac{2\pi\Theta}{\lambda}} - 2k(1-k^2) . \sin\overline{\alpha+\phi} . \cos\overline{\xi + \frac{2\pi\Theta}{\lambda}}$$

$$-k^2(1-k^2)\cos\overline{\alpha+\phi} . \sin\overline{\xi + \frac{4\pi\Theta}{\lambda}} + k(1-k^2) . \sin\overline{\alpha+\phi} . \cos\overline{\xi + \frac{4\pi\Theta}{\lambda}} .$$

Vibration parallel to $o_1 o_2 =$

$$-k^2(1-k^2)\cdot\sin\overline{\alpha+\phi}\cdot\sin\xi+k(1-k^2)\cdot\cos\overline{\alpha+\phi}\cdot\cos\xi$$

$$+4k^2\cdot\sin\overline{\alpha+\phi}\cdot\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}}-2k(1-k^2)\cos\overline{\alpha+\phi}\cdot\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}}$$

$$+(1-k^2)\cdot\sin\overline{\alpha+\phi}\cdot\sin\overline{\xi+\frac{4\pi\Theta}{\lambda}}+k(1-k^2)\cdot\cos\overline{\alpha+\phi}\cdot\cos\overline{\xi+\frac{4\pi\Theta}{\lambda}}.$$

To avoid unnecessary generalities we will suppose the plates crossed, or $\alpha = 90^0$: which gives

Vibrations parallel to $Oo_1 =$

$$-(1-k^2)\cdot\sin\phi\cdot\sin\xi+k(1-k^2)\cdot\cos\phi\cdot\cos\xi$$

$$-4k^2\sin\phi\cdot\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}}-2k(1-k^2)\cos\phi\cdot\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}}$$

$$+k^2(1-k^2)\sin\phi\cdot\sin\overline{\xi+\frac{4\pi\Theta}{\lambda}}+k(1-k^2)\cos\phi\cdot\cos\overline{\xi+\frac{4\pi\Theta}{\lambda}}.$$

Vibrations parallel to $o_1 o_2 =$

$$-k^2(1-k^2)\cos\phi\cdot\sin\xi-k(1-k^2)\cdot\sin\phi\cdot\cos\xi$$

$$+4k^2\cos\phi\cdot\sin\overline{\xi+\frac{2\pi\Theta}{\lambda}}+2k(1-k^2)\sin\phi\cdot\cos\overline{\xi+\frac{2\pi\Theta}{\lambda}}$$

$$+(1-k^2)\cos\phi\cdot\sin\overline{\xi+\frac{4\pi\Theta}{\lambda}}-k(1-k^2)\sin\phi\cdot\cos\overline{\xi+\frac{4\pi\Theta}{\lambda}}.$$

The efficient vibration, or that perpendicular to AP_2, will be found by multiplying the former of these by cos ϕ, the latter by sin ϕ, and taking their sum. Thus we have

$$-\frac{1-k^4}{2}\sin 2\phi\cdot\sin\xi+k(1-k^2)\cos 2\phi\cdot\cos\xi$$

$$-2k(1-k^2)\cos 2\phi \,.\, \overline{\cos \xi + \frac{2\pi\Theta}{\lambda}}$$

$$+\frac{1-k^4}{2}\sin 2\phi \,.\, \overline{\sin \xi + \frac{4\pi\Theta}{\lambda}} + k(1-k^2)\cos 2\phi \,.\, \overline{\cos \xi + \frac{4\pi\Theta}{\lambda}}.$$

The coefficient of $\sin \xi$ is

$$-\frac{1-k^4}{2}\sin 2\phi + 2k(1-k^2)\cos 2\phi \,.\, \sin\frac{2\pi\Theta}{\lambda}$$

$$+\frac{1-k^4}{2}\sin 2\phi \,.\, \cos\frac{4\pi\Theta}{\lambda} - k(1-k^2)\cos 2\phi \,.\, \sin\frac{4\pi\Theta}{\lambda}.$$

The coefficient of $\cos \xi$ is

$$k(1-k^2)\cos 2\phi - 2k(1-k^2)\cos 2\phi \,.\, \cos\frac{2\pi\Theta}{\lambda}$$

$$+\frac{1-k^4}{2}\sin 2\phi \,.\, \sin\frac{4\pi\Theta}{\lambda} + k(1-k^2)\cos 2\phi \,.\, \cos\frac{4\pi\Theta}{\lambda}.$$

The sum of their squares is, after all reductions,

$$\overline{1-k^2}|^2 \,.\, \sin^2\frac{\pi\Theta}{\lambda} \,.\, \left\{4k \,.\, \cos 2\phi \,.\, \sin\frac{\pi\Theta}{\lambda} - 2\,.\,(1+k^2)\,.\,\sin 2\phi \,.\, \cos\frac{\pi\Theta}{\lambda}\right\}^2.$$

And restoring the factor $\frac{c^2}{(1+k^2)^4}$, we have for the brightness

$$c^2 \,.\, \left(\frac{1-k^2}{1+k^2}\right)^2 \,.\, \sin^2\frac{\pi\Theta}{\lambda}\left\{\frac{4k}{1+k^2}\cos 2\phi \,.\, \sin\frac{\pi\Theta}{\lambda} - 2\,.\,\sin 2\phi \,.\, \cos\frac{\pi\Theta}{\lambda}\right\}^2.$$

1. If we make $\tan \chi = \frac{1+k^2}{2k}\tan 2\phi$, this expression becomes

$$c^2 \,.\, \frac{(1-k^2)^2}{(1+k^2)^4}(16k^2 \,.\, \cos^2 2\phi + 4\,.\,\overline{1+k^2}|^2 \,.\, \sin^2 2\phi) \,.\, \sin^2\frac{\pi\Theta}{\lambda} \,.\, \sin^2\left(\frac{\pi\Theta}{\lambda} - \chi\right).$$

This vanishes, or there are black lines, when $\sin^2\frac{\pi\Theta}{\lambda} = 0$, or when

$$\frac{\pi\Theta}{\lambda} = 0, \text{ or } = \pi, \text{ or } = 2\pi, \text{ \&c.}$$

This indicates a series of dark circles, whose diameters are the same as those of the circles seen with either of the plates singly.

2. The expression also vanishes when

$$\sin^2\left(\frac{\pi\Theta}{\lambda} - \chi\right) = 0,$$

or when

$$\frac{\pi\Theta}{\lambda} = \chi, \quad = \pi + \chi, \quad = 2\pi + \chi, \quad \&c.$$

Now when 2ϕ increases from 0 to 90°, from 90° to 180°, &c. χ also increases from 0 to 90°, from 90° to 180°, &c.: consequently χ will never differ much from 2ϕ. So that the expression vanishes when

$$\frac{\pi\Theta}{\lambda} = 2\phi, \quad = \pi + 2\phi, \quad = 2\pi + 2\phi, \quad \&c. \text{ nearly.}$$

In the curve defined by this equation, it is plain that Θ increases continually as ϕ increases, and consequently θ increases continually as ϕ increases. The curve therefore is a spiral in such a position that if we look at a point above the center, and follow the curve towards the right-hand, the radius vector continually increases.

3. Now if we increase ϕ by 90°, or 180°, or 270°, we get the same values for χ, increased by π, or 2π, or 3π. Consequently the values of the radius vector, at a point of the dark line, are found by making $\frac{\pi\Theta}{\lambda}$ equal

in the first of these to $\pi + \chi$, $2\pi + \chi$, $3\pi + \chi$, &c.
in the second to $2\pi + \chi$, $3\pi + \chi$, $4\pi + \chi$, &c.
in the third to $3\pi + \chi$, $4\pi + \chi$, $5\pi + \chi$, &c.

These are evidently the same series of values as that found with the original value of ϕ. That is, if we draw four radii vectores of equal length at angles of 90°, and if one of these terminate in a point of the dark curve,

the others will also terminate in points of the dark curve. This condition can be reconciled with the spiral form of the curve, only by supposing that the curve consists of four similar spirals, each of which is turned 90° from the position of that adjacent to it. The general form of the curves will be therefore four spirals, in positions differing by 90°, and all turned the same way, intersecting a series of circles. This very remarkable form is precisely the form given by observation.

4. The intersection of the spirals and circles is found by making both the equations

$$\sin^2 \frac{\pi\Theta}{\lambda} = 0 \text{ and } \sin^2\left(\frac{\pi\Theta}{\lambda} - \chi\right) = 0,$$

to hold at the same time. This gives

$$\chi = 0, \text{ or } = \pi, \text{ or } = 2\pi, \text{ \&c.:}$$

and consequently

$$\phi = 0, \text{ or } = \frac{\pi}{2}, \text{ or } = \pi, \text{ or } = \frac{3\pi}{2}.$$

That is, the intersections of the spirals and circles will all lie in the lines through the center parallel and perpendicular to the plane of reflexion. This is exactly true in the experiment.

5. And since the successive circles require values of $\frac{\pi\Theta}{\lambda}$ successively increasing by π, and the successive points of the spirals on the same radius vector also require values of $\frac{\pi\Theta}{\lambda}$ successively increasing by π, every circle will be intersected by the spirals in the lines above mentioned, and therefore every circle will be intersected at every quadrant. This is verified by experiment.

6. If we consider the parts near the center and not in the circumference of one of the circles, the angle ϕ' corresponding to a dark point will be nearly

$$\frac{\pi\Theta'}{2\lambda}, \text{ or } \frac{\pi\Theta'}{2\lambda} + 90^0, \text{ \&c.}$$

where Θ' is the value of Θ corresponding to $\theta = 0$,

$$\text{or} \quad \Theta' = \frac{eT}{\lambda\pi}.$$

Consequently,

$$\phi' = \frac{eT}{2\lambda^2}, \quad \text{or} \quad \frac{eT}{2\lambda^2} + 90^0, \text{ \&c.}$$

But $\frac{eT}{\lambda^2}$ is the angle through which the analyzing plate must be turned to the left to see the dark spot with the right-handed plate alone. Consequently the dark cross in which the spirals originate is inclined, (the upper part to the right), by an angle half as great as the angle through which the analyzing plate must be turned (the upper part to the left) to see the dark spot with the right-handed plate only. This appears to agree with experiment.

7. In the whole of this we have supposed the right-handed plate to be nearest to the polarizing plate. If the combination be turned, with the left-handed plate nearest to the polarizing plate, we must change the sign of k. This changes the sign of χ (that of ϕ being supposed the same); and it will very easily be seen that in consequence of this change, the direction and position of the spirals will be exactly inverted. This is found to be true.

8. When k is small, the expression for the brightness becomes small if $\sin^2 2\phi = 0$, that is, if $\phi = 0$, or $= 90^0$, &c. This accounts for the dark brushes seen parallel and perpendicular to the plane of reflexion at a great distance from the center.

9. When ϕ is $> 0 < 45^0$, χ is $> 2\phi$: when ϕ is $> 45^0 < 90^0$, χ is $< 2\phi$. Observing that the spirals intersect the circles when $\phi = 0$, $= 90^0$, &c., it is easily seen from this that they cut them at an angle rather greater than the angle at which a uniform spiral would cut them. This seems to be observable in the experiment.

10. The peculiar vividness of the colours appears to be explained by this consideration. The dark curves are defined by making the expression for the brightness to vanish totally: whereas in some other cases (as in investigation VII.) the dark curves are defined by making the expression for the brightness a minimum. Here then the colours are much less diluted with undestroyed white light than in investigation VII. And in the lines parallel and perpendicular to the plane of reflexion, $\chi=0$, and the expression for the brightness is $c^2 . \frac{(1-k^2)^2}{(1+k^2)^4} 16k^2 . \sin^4 \frac{\pi\Theta}{\lambda}$. Here then is no sensible light for a considerable distance nearer to and farther from the center than the point corresponding to $\frac{\pi\Theta}{\lambda} = n\pi$: the colours are less mixed, and are therefore more vivid than perhaps in any other phenomenon of polarization. And the greatest brightness in these lines, if k be not very small, is four times as great as the greatest brightness in the lines making angles of 45^0 with them (as will be seen on making $\phi=45^0$ in the expression for the brightness). Experimentally the colours are brightest in the lines parallel and perpendicular to the plane of reflexion.

It is almost unnecessary to point out to the reader that none of the peculiarities of these appearances would exist if k were either 0 or 1, that is, if the light were either plane-polarized or circularly-polarized. None of the expressions however would be altered, if, instead of k we put $\frac{1}{k}$: that is, it is indifferent which ray we suppose to have the major axis of its ellipse parallel to the principal plane.

From the agreement between the observed and the calculated appearances, I think there is little doubt that the nature of the

light in the two rays of quartz is such as I have described. I do not mean to exclude the possibility of supposing that the form of neither wave (in the construction for determining the course of the rays) is exactly spherical or exactly spheroidal: provided the difference of the forms be nearly the same as that of a sphere and a spheroid. Nor do I mean to assert that each elliptically-polarized ray consists exactly of two plane-polarized rays following each other at the interval of one-fourth of an undulation: or that the ratio of the two axes in the two rays is exactly the same. But I conceive it to be perfectly certain that the general character of the light is such as is stated in my hypotheses.

I have not made any calculations upon other suppositions, but I can hardly imagine that any other would represent the phenomena to such extreme accuracy. I am not so much struck with the accounting for the continued dilatation of circles, and the general representation of the form of spirals, as with the explanation of the minute deviations from symmetry, as when circles become almost square, and crosses are inclined to the plane of polarization. And I believe that any one who shall follow my investigations and imitate my experiments, will be surprized at their perfect agreement.

There is one relation between the construction for determining the course of the rays, and the nature of the rays, which deserves (I think) particular attention. It is that (comparing the rays of quartz with those of any other crystal) a change in the nature of the ray is accompanied with an interruption of continuity. The *nappes* of the wave surfaces are absolutely separated. This is not the case in the common construction for uniaxal crystals, nor in Fresnel's construction for biaxal crystals.

There may possibly be a connection of the same kind as that between the change from partial reflexion to total reflexion within glass, and the accompanying change from plane polarized light to elliptically-polarized light. The cases are at least thus far analogous, that the change in the light and the interruption of continuity go together. But we are so much in the dark respecting the physical constitution of quartz, that we cannot at present go farther.

It might have been desirable to verify my suppositions by more direct experiments on the separate rays of quartz. I can only plead that the duties of my office have not allowed me the necessary time. They would (under all circumstances) have been much more troublesome than those which I have the honor of laying before the Society: and I do not think that they would have been more satisfactory. The appearances presented by depolarization are admirably adapted to the discovery of the most delicate differences in the nature and course of rays. The same want of time I hope will be allowed as an excuse for the want of accurate measures*: without which no theory, however satisfactory in general explanations, can be considered as firmly established.

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Dec. 30, 1830.

* It is much to be wished that the rings and spirals exhibited by quartz may be accurately measured, their diameters in different directions ascertained, and a comparison with theory instituted, by means of Biot's measures of the doubly refractive energy of quartz. Should the observations and the theory disagree, it would shew, either that there is some latent error in the theory, or that the difference of curvature of the sphere and spheroid near their vertices is not the same as that which is inferred from Huyghen's construction, modified as above.

IV. *On the Resolution of Algebraical Equations.*

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INTRODUCTION.

THE researches of Lagrange on that part of Pure Analysis, which forms the subject of the present Memoir, have been followed up with considerable success by many foreign Mathematicians, amongst whom M. Augustin Cauchy deserves to be particularly distinguished; indeed the extensive use of the theorem of Lagrange in Physical Astronomy, had turned the attention of Analysts to consider more intimately the nature of that series, the conditions of its convergence, the root which it particularly represents, &c. I have referred to as many papers on this subject, scattered through the Memoirs of the French Institute, the Journal of the Polytechnic School, and the Annales de Mathematiques, &c. as I conveniently could. I do not find that in the point of view in which this subject is here exhibited, I have been anticipated in any of the articles above referred to, a point on which it is necessary to be doubtful, without actual reference, from the great number of persons who have been recently, and are at present, engaged in extending the limits of Analysis.

I have supposed the given equation to be such as to contain neither negative nor fractional powers of the unknown quantity x; if such should enter any proposed equation as $\phi(x)=0$, we need only put $x=a+z$, and consider z as the unknown quantity, since $\phi(a+z)$ may always be expanded according to the positive and integer powers of z, when no particular value is assigned to a, this is therefore to be understood, unless where the contrary is expressed.

Suppose the root of an equation $\phi(x)=0$ is sought, the following simple rule which I have proved and applied in Section 1. will give it with great facility.

"Divide the given equation by x, take the Nap. log. of the quotient by means of the formula

$$\text{l.}\overline{1+z}=z-\frac{z^2}{2}+\frac{z^3}{3}, \text{ \&c.}$$

take the coefficient of the first negative power of x in this logarithmic expansion: this, with its sign changed, is the root of the proposed equation."

If the proposed equation were of n dimensions, it has n roots, and it is natural to enquire which root is given by the preceding method. I have in the same Section shewn that it analytically gives a result which comprehends all the roots, but that arithmetically it gives the least root.

If, instead of the root of an equation, any function $f(x)$ of the root should be required, there is given in Section (2) for this purpose, a rule nearly as simple as the above; namely,

"Take the same Naperian log. as before. Multiply it by the derived function $f'(x)$, the coefficient of the first negative

power of x, with its sign changed, will be the required function of the root; minus the same function of 0."

In Section 3, there is given a method equally simple with the former ones, to obtain the sum of any specified number of the roots; also the sum of any given function of a specified number of the roots.

In this Section it is also shewn how to find the mth least root of an equation, or any function of it, to which are annexed some remarks on that relation of imaginary quantities, which corresponds to the relation of greater and less in real quantities.

As the sum of m roots exceeds that of $\overline{m-1}$ by one root, it is clear that by this method we can get all the roots of the equation, as well as any function of any root.

The principles laid down in the first three Sections are applied in the fourth to the deduction of several theorems of analysis; the theorems of Laplace and Lagrange are simple and almost immediate consequences: a theorem somewhat similar to Lagrange's, which M. Cauchy gives in (Vol. IX. *Memoirs of the Institute,*) for the sum of any function of all the roots of an equation, I have here shewn holds true for any specified number of the roots as for the whole, though the series only terminates in the latter case, and under particular conditions.

There is also a theorem given by Burmann, which is of great use in transforming series, but the ordinary demonstration is very long, and may be seen in the notes at the end of Vol. III. Lacroix, *Diff. Calc.* but in the present method it follows in a few lines.

When we revert a proposed series, or find the root of an algebraic equation in a series, the law of the latter is not

readily visible: the same Section contains the expression of this law.

There is besides in this Section, a remarkable expression for the correction to be applied to the root of an equation, when to the equation itself there is added a small term.

Next in order, the important subject of the Limits of Lagrange's series is considered.

Section V. treats on Definite Integrals: in this Section the root of any proposed equation, any function of the root, &c., are all represented by means of Definite Integrals; and we here see the Analytical use of the multiplicity of the values of certain transcendent functions, when the quantity under the sign of such a function should be the same at both the limits of an integral.

The last Section contains various points of Analysis, connected with the present subject, which could not conveniently be brought into any of the others.

SECTION I.

To find the root of any equation $\phi(x)=0$, which contains only positive and integer powers of x.

Divide the equation by x, ***take the Nap. log. of the quotient; the coefficient of the first negative power of*** x, ***with its sign changed, is the root of the equation.***

For example, in the quadratic equation

$$x^2+ax+b=0,$$

divide by x, and take the log. of the quotient,

$$\text{i. e. l.}\left(a+x+\frac{b}{x}\right),\ \text{or l.}\,a+\text{l.}\left(1+\frac{x+\frac{b}{x}}{a}\right),$$

by the preceding rule the root should be therefore the coefficient of $\frac{1}{x}$ in

$$-\frac{x+\frac{b}{x}}{a}+\tfrac{1}{2}\cdot\frac{\left(x+\frac{b}{x}\right)^2}{a^2}-\tfrac{1}{3}\cdot\frac{\left(x+\frac{b}{x}\right)^3}{a^3}+\text{\&c.},$$

and selecting the coefficient of $\frac{1}{x}$ which enters only in the 1st, 3rd, 5th, &c. terms, we get for the required root

$$-\left\{\frac{b}{a}+\frac{b^2}{a^3}+\frac{4}{2}\cdot\frac{b^3}{a^5}+\frac{6.5}{2.3}\cdot\frac{b^4}{a^7}+\frac{8.7.6}{2.3.4}\cdot\frac{b^5}{a^9},\ \text{\&c.}\right\}.$$

as it evidently is, since this series represents the expansion of

$$-\frac{a}{2}+\left(\frac{a^2}{4}-b\right)^{\frac{1}{2}}.$$

This instance of the application of the present method is sufficient to shew its nature, but as we may frequently facilitate the operation by a slight previous transformation of the proposed equation, I have added a few more examples.

To find z in the equation

$$az^n + z - b = 0,$$

in this case, the rule may be directly applied; but the result may be obtained, rather more simply, by putting $z - b = x$.

The equation becomes

$$a(x+b)^n + x = 0,$$

therefore by the rule

$$x = \text{coefficient of } \frac{1}{x} \text{ in } -1.\left\{1 + \frac{a}{x}.(x+b)^n\right\}$$

$$= \text{coefficient of } \frac{1}{x} \text{ in } -\frac{a}{x}(x+b)^n + \tfrac{1}{2}.\frac{a^2}{x^2}(x+b)^{2n} - \tfrac{1}{3}.\frac{a^3}{x^3}.(x+b)^{3n} + \&c.$$

$$= -ab^n + 2n.b^{2n-1}.\frac{a^2}{2} - \frac{3n.\overline{3n-1}}{1.2}.b^{3n-2}.\frac{a^3}{3} + \&c.$$

As another example, suppose $a + z = 1.(z)$ to find z,

$$\text{or } z - e^a.e^z = 0.$$

Hence by the rule

$$z = \text{coefficient of } \frac{1}{z} \text{ in } -1.\left(1 - e^a.\frac{e^z}{z}\right)$$

$$= \text{coefficient of } \frac{1}{z} \text{ in } e^a.\frac{e^z}{z} + \frac{e^{2a}}{2}.\frac{e^{2z}}{z^2} + \frac{e^{3a}}{3}.\frac{e^{3z}}{z^3} + \&c.$$

$$= e^a + \tfrac{1}{2}.\frac{2}{1}.e^{2a} + \tfrac{1}{3}.\frac{3^2}{1.2}.e^{3a} + \&c.$$

When the given equation contains fractional or negative powers of x, put (as was before observed) $x = z + a$, and considering z as the unknown quantity, we may directly apply the method.

The rule given in this section may be thus proved. Suppose $\phi(x)$ to be resolved into its simple factors; i. e.

$$\phi(x) = C.(x-\alpha).(x-\beta).(x-\gamma)$$

$$\text{then } \frac{\phi(x)}{x} = C'\left(1-\frac{\alpha}{x}\right).\left(1-\frac{x}{\beta}\right).\left(1-\frac{x}{\gamma}\right)\ldots \&c.$$

$$\text{where } C' = C.(-\beta).(-\gamma).\&c.$$

$$\therefore \text{ l.}\frac{\phi(x)}{x} = \text{l. } C' + \text{l.}\left(1-\frac{\alpha}{x}\right) + \text{l.}\left(1-\frac{x}{\beta}\right) + \text{l.}\left(1-\frac{x}{\gamma}\right) + \&c.$$

of which the only term which contains negative powers of x, is

$$\text{l.}\left(1-\frac{\alpha}{x}\right) \text{ or } -\left\{\frac{\alpha}{x} + \tfrac{1}{2}.\frac{\alpha^2}{x^2} + \tfrac{1}{3}.\frac{\alpha^3}{x^3} + \&c.\right\},$$

the coefficient of $\frac{1}{x}$ therefore in l. $\frac{\phi(x)}{x}$ is $-\alpha$, where α is a root of the equation $\phi(x) = 0$.

Supposing the equation of n dimensions, which of its roots is given by this method? The result analytically comprehends all the roots, but arithmetically it gives only the least root. Let us recall, for example, the quadratic equation $x^2 + ax + b = 0$; and, supposing α, β, to be the roots; $-a = \alpha + \beta$, $b = \alpha\beta$; and substituting these values in the expression for the root given at the beginning of this Section, we get

$$\text{root} = \frac{\alpha\beta}{\alpha+\beta} + \frac{\alpha^2\beta^2}{(\alpha+\beta)^3} + \frac{4}{2}.\frac{\alpha^3\beta^3}{(\alpha+\beta)^5} + \frac{6.5}{2.3}.\frac{\alpha^4\beta^4}{(\alpha+\beta)^7} + \&c.$$

and expanding each term of this series according to the descending powers of β, the value of the root is

$$\begin{aligned}
&\alpha\left\{1 - \frac{\alpha}{\beta} + \frac{\alpha^2}{\beta^2} - \frac{\alpha^3}{\beta^3} + \frac{\alpha^4}{\beta^4} \text{ \&c.}\right.\\
&+\frac{\alpha^2}{\beta}\left\{1 - 3.\frac{\alpha}{\beta} + \frac{3.4}{1.2}\cdot\frac{\alpha^2}{\beta^2} - \frac{4.5}{1.2}\cdot\frac{\alpha^3}{\beta^3} \text{ \&c.}\right.\\
&+\frac{4}{2}\cdot\frac{\alpha^3}{\beta^3}\left\{1 - 5.\frac{\alpha}{\beta} + \frac{5.6}{1.2}\cdot\frac{\alpha^2}{\beta^2} - \text{\&c.}\right.\\
&+\frac{6.5}{2.3}\cdot\frac{\alpha^4}{\beta^4}\left\{1 - 7\frac{\alpha}{\beta} + \text{\&c.}\right.\\
&+\frac{8.7.6}{2.3.4}\cdot\frac{\alpha^5}{\beta^5}\left\{1 - \text{\&c.}\right.\\
&\text{\&c.} \qquad \text{\&c.}
\end{aligned}$$

All the terms here mutually strike out with the exception of the first α; but since α and β are similarly involved in the given series, it follows that if we expanded according to the descending powers of α, all the terms would strike out except β; thus this series analytically represents α or β indifferently; but if we stop at the nth term in the former case, the error is of the order $\left(\frac{\alpha}{\beta}\right)^n$, and in the latter, of $\left(\frac{\beta}{\alpha}\right)^n$; and if $\alpha < \beta$, the former is very small, and the latter great; the series therefore, arithmetically, gives the least root (abstracting from its sign); and generally, whatever is the proposed equation, the series expressing the root is manifestly a symmetrical function of all the roots, and therefore does not *analytically* express one more than another, but comprehends alike all the roots; but as each term may be expanded in a con-

verging form when we make use of the ascending powers of the least root, the given series arithmetically designates the least root. And thus it is that Lagrange's series which, as it will be hereafter seen, coincides with that obtained by the present method, represents arithmetically the least root of the proposed equation.*

SECTION II.

To find any function† $f(x)$ of the root of a given equation $\phi(x) = 0$.

From $\mathrm{f}(0)$ ***subtract the coefficient of*** $\frac{1}{x}$ ***in*** $\mathrm{f}'(x) \,.\, l. \frac{\phi(x)}{x}$; $\mathrm{f}'(x)$ ***being the derived function or differential coefficient of*** $\mathrm{f}(x)$, ***the remainder will be the required function.***

After the examples given in the former Section, many will not be necessary, in illustration of the present principle; we shall take but one, viz. to find the value of x^n when $x = c \,.\, \epsilon^x$, n being positive.

The required value of x^n, by the above principle, is the coefficient of $\frac{1}{x}$ in $-n x^{n-1} \mathrm{l}. (1 - \frac{c}{x} . \epsilon^x)$; and expanding the log. and ob-

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* When the first power of x does not enter the given equation, put $x = a + z$, so as to introduce a first power of z; and then, let z be treated as the unknown quantity.

† $f(x)$ is supposed to contain only positive and integer powers of x; should it be otherwise, we have only to put $x = a + z$; the same is to be observed of $\phi(x)$.

serving that the first $n-1$ terms may be rejected as not containing any negative powers of x, we get

$$x^n = n \times \text{coefficient of } \frac{1}{x} \text{ in } \left\{\frac{c^n \cdot \epsilon^{nx}}{nx} + \frac{c^{n+1} \cdot \epsilon^{\overline{n+1} \cdot x}}{\overline{n+1} \cdot x^2} + \&c.\right\},$$

that is,

$$x^n = c^n + n c^{n+1} + \frac{n \cdot \overline{n+2}}{1.2} \cdot c^{n+2} + \frac{n \cdot \overline{n+3^2}}{1.2.3} \cdot c^{n+3} + \&c.$$

This principle evidently includes that given in the former section as a particular case, and admits of a proof nearly as simple, viz.

Put $\phi(x)$ in the same form $C \cdot \overline{x-\alpha} \cdot \overline{x-\beta} \cdot \overline{x-\gamma}$ &c.

$$\text{and } \therefore \text{ l.} \frac{\phi(x)}{x} \text{ is l. } C' + \text{l.} \overline{1-\frac{\alpha}{x}} + \text{l.} \overline{1-\frac{x}{\beta}} + \text{l.} \overline{1-\frac{x}{\gamma}} \&c.$$

the part of which containing negative powers of x is

$$-\left\{\frac{\alpha}{x} + \frac{1}{2} \cdot \frac{\alpha^2}{x^2} + \frac{1}{3} \cdot \frac{\alpha^3}{x^3} \&c.\right\}.$$

Hence $\frac{\alpha^n}{n}$ is evidently the coefficient of $\frac{1}{x^n}$ in $-\text{l.} \frac{\phi(x)}{x}$,

$$\text{and } \therefore \alpha^n = \text{coefficient of } \frac{1}{x} \text{ in } -n x^{n-1} \text{ l.} \frac{\phi(x)}{x}.$$

Let $f(\alpha)$ be any function of α, as $f(0) + A\alpha + B\alpha^2 + C\alpha^3$ &c. putting for n 1, 2, 3 &c. successively, it follows that

$$f(\alpha) = f(0) + \text{coefficient of } \frac{1}{x} \text{ in } -\{A + 2Bx + 3Cx^2 \&c.\} \cdot \text{l.} \frac{\phi(x)}{x}$$

$$= f(0) - \text{coefficient of } \frac{1}{x} \text{ in } f'(x) \cdot \text{l.} \frac{\phi(x)}{x}.$$

SECTION III.

To find the sum of any specified number (m) of the roots of an equation.

Divide the given equation by x^m, ***take the Nap. log. of the quotient; the coefficient of*** $\frac{1}{x}$, ***with its sign changed, will be the required sum of*** m ***roots.***

Ex. (1). $x^2 + ax + b = 0$, to find the sum of two roots, take the coefficient of $\frac{1}{x}$ in $-\text{l.}\left(1 + \frac{a}{x} + \frac{b}{x^2}\right)$ viz. $-a$, which is evidently the sum of the two roots.

Ex. (2). $x^2 - ax^n - b = 0$, to find the sum of two roots. By the above rule, that sum

$$= \text{coefficient of } \frac{1}{x} \text{ in } -\text{l.}\left(1 - \frac{ax^n + b}{x^2}\right)$$

$$= \text{coefficient of } \frac{1}{x} \text{ in } \frac{ax^n + b}{x^2} + \frac{(ax^n + b)^2}{2x^4} + \frac{(ax^n + b)^3}{3x^6} + \&c.$$

Hence, when n is even, this quantity is nothing which agrees with truth, since the roots of the proposed equation are then of the form $\pm\sqrt{a}$.

But when n is odd, the terms of this series which involve $\frac{1}{x}$, are at n places distance from each other, and begin at the $\frac{n+1}{2}$ term, its value therefore is

$$ab^{\frac{n-1}{2}} + \frac{\overline{3n-1}.\overline{3n-3}}{2.4}.\frac{a^3}{3}.b^{\frac{3n-3}{2}} + \frac{\overline{5n-1}.\overline{5n-3}.\overline{5n-5}.\overline{5n-7}}{2.4.6.8}.\frac{a^7}{7}.b^{5\frac{n-7}{2}} \&c.$$

When the term involving x^m is wanting in the proposed equation, then, instead of dividing immediately by x^m, put $x = z + a$, and the difficulty of having no term free from the unknown quantity in the quotient (on which depends the expansion of the log.) will be avoided.

To prove the rule given in this Section, suppose $a_1, a_2 \ldots a_m, a_{m+1} \ldots a_n$ to be the n roots of the equation $\phi(x) = 0$.

Hence $\phi(x) = C.(x-a_1).(x-a_2)\ldots\ldots(x-a_m)\,(x-a_{m+1})\ldots\ldots(x-a_n)$;

$$\therefore \frac{\phi(x)}{x^m} = C.\left(1-\frac{a_1}{x}\right)\left(1-\frac{a_2}{x}\right)\ldots\ldots\left(1-\frac{a_m}{x}\right)$$

$$\times\left(1-\frac{x}{a_{m+1}}\right)\left(1-\frac{x}{a_{m+2}}\right)\ldots\ldots\left(1-\frac{x}{a_n}\right),$$

and taking the log. of both sides, it is manifest that

$$a_1 + a_2 + \ldots a_m = \text{coefficient of } \frac{1}{x} \text{ in } -\text{l.}\frac{\phi(x)}{x^m}.$$

This method then will give us very simply the sum of any proposed number (m) of the roots of the given equation; but since there may be various combinations of m roots made from the n roots of the given equation, which combination or group of m roots is that given by the present rule? The answer is ***analytically***, it gives any possible group of the m roots, but ***arithmetically*** the m least: that it analytically gives any is obvious, since the resulting expression is manifestly a symmetrical function of all the roots, and therefore cannot analytically represent any one combination of m roots more than any other. That it gives the sum of the m least roots may be thus shewn.

Suppose, first, that the sum of two roots of the equation $\phi(x) = 0$ is obtained by the present method, and that $a_1 + a_2$ is the particular combination which it gives.

Hence $a_1 + a_2 =$ coefficient of $\frac{1}{x}$ in $-1.\frac{\phi(x)}{x^2}$.

Now since a_1 is a root, therefore $\phi(x)$ is of the form $\overline{x-a_1}.P$;

therefore $a_1 + a_2 =$ coefficient of $\frac{1}{x}$ in $-1.\overline{1-\frac{a_1}{x}} - 1.\frac{P}{x}$

$= a_1 -$ coefficient of $\frac{1}{x}$ in $1.\frac{P}{x}$,

but the coefficient of $\frac{1}{x}$ in $-1.\frac{P}{x}$ is the least root of the equation $P=0$ (by Section 1).

Hence a_2 must be the least of the quantities: a_2, a_3, a_4, &c.: by similar proof a_1 must be the least of a_1, a_3, a_4, &c.; and therefore a_1 and a_2 are the two least of a_1, a_2, a_3, &c., that is, they are the two least roots.

In the same manner if by the present method we get the sum of three roots $a_1 + a_2 + a_3$, then putting $\phi(x) = \overline{x-a_1}.\overline{x-a_2}.P'$, we have

$$-1.\frac{\phi x}{x^3} = -1.\left(1-\frac{a_1}{x}\right) - 1.\left(1-\frac{a_2}{x}\right) - 1.\frac{P'}{x},$$

and taking the coefficients of $\frac{1}{x}$ at both sides by the theorems already established, we have

$a_1 + a_2 + a_3 = a_1 + a_2 +$ least root of the equation $P'=0$;

therefore a_3 is the least of the quantities a_3, a_4, a_5, &c.

Similarly a_2 ..... a_2, a_4, a_5, &c.

a_1 ..... a_1, a_4, a_5, &c.

therefore a_1, a_2, a_3, are the 3 least roots of the equation: thus it appears in general that this method gives the sum of the m least roots.

In comparing thus the magnitude of the roots, we go on the hypothesis of their reality. When the roots are imaginary however, a similar order may be supposed to subsist; thus $-a+\sqrt{\frac{a^2}{4}-b}$ is a less numerical root than $-a-\sqrt{\frac{a^2}{4}-b}$ (abstracting from the sign) when $\frac{a^2}{4} > b$, it holds therefore the same rank when $\frac{a^2}{4} < b$.

Since the whole theory of imaginary quantities results from an extension of the properties of real quantities, we have this advantage resulting from the theorem above proved, that it establishes an order when quantities are some real and some imaginary—analogous to the relation of greater and less amongst real quantities.

To find the mth least root of a proposed equation $\phi(x) = 0$.

From the coefficient of $\frac{1}{x}$ in l. $\frac{\phi(x)}{x^{m-1}}$ subtract the coefficient of $\frac{1}{x}$ in l. $\frac{\phi(x)}{x^m}$, the remainder will be the mth least root. This theorem is an immediate consequence of what has been already proved in this Section.

To find the sum of any function $f(a_1)\, f(a_2)$ &c. of the m least roots.

From $mf(0)$ subtract the coefficient of $\frac{1}{x}$ in $f'(x)$. l. $\frac{\phi(x)}{x^m}$, the remainder will be the required quantity.

The proof of this theorem is similar to that given for a function of one root, it will be unnecessary therefore to add it.

To find the value of any function of the mth least root.

Find, by the last theorem, the sum of the values of that function for the m least roots, and also for the $(m-1)$ least roots, and take the difference.

SECTION IV.

In this Section we shall make a few applications of the present method, to shew with what facility it gives various theorems of Analysis.

Lagrange's Theorem.

Let $z = a + hF(z)$ to find $f(z)$ any function of z; put $z = x + a$, the equation becomes $x = hF(a+x)$; to find $f(x+a)$, apply the rule in Section (2).

Hence

$$f(z) \text{ or } f(x+a) = f(a) - \text{ coefficient of } \frac{1}{x} \text{ in } f'(a+x)\,\mathrm{l.}\left(1 - \frac{hF\overline{a+x}}{x}\right)$$

$$= f(a) + \text{ coefficient of } \frac{1}{x} \text{ in}$$

$$\left\{\frac{h}{x}f'(a+x).F(a+x) + \tfrac{1}{2}.\frac{h^2}{x^2}f'\overline{a+x}.\overline{Fa+x}^2 \text{ \&c.}\right\}.$$

Now if we consider $f'(a+x)F(a+x)$, $f'(a+x)\overline{F(a+x)}^2$, &c. as so many separate functions of $a+x$, and expand them by Taylor's Theorem, it is visible that the above equation becomes

$$f(z) = f(a) + hf'(a).F(a) + h^2.\frac{(f'(a).\overline{Fa}^2)'}{1.2} + h^3\frac{(f'a.\overline{Fa}^3)''}{1.2.3} \text{ \&c.}$$

Laplace's Theorem.

Let $z = F\{a + h\phi(z)\}$ to find $f(z)$.

Put $z = F(a+u)$ the equation becomes $u = h\phi . F(a+u)$, to find $fF(a+u)$, or $f(z)$ apply the rule in Section (2).

Hence

$$f(z) = fF(a) - \text{coefficient of } \frac{1}{u} \text{ in } f'F(a+u)\,\text{l.}\left\{1 - h\frac{\phi . F(a+u)}{u}\right\}$$

and expanding the log. as in the last case, it evidently gives

$$f(z) = fF(a) + hf'F(a).\phi F(a) + \frac{h^2}{1.2}(f'F(a).\overline{\phi F}a^2)' + \&c.$$

Burmann's Theorem consists in this, that if z and u are two functions of x which vanish together, and X any other function, then

$$\frac{d^n X}{du^n} = d^{n-1}\frac{\left\{\frac{dX}{dz}.\left(\frac{z}{u}\right)^n\right\}}{dz^{n-1}}$$

putting z and $u = 0$ after the differentiations.

In fact, if we form the equation $u = hX^{\frac{1}{n+1}}$, the value of u is

coefficient of $\frac{1}{u}$ in $-\text{l.}\left(1 - h.\frac{X^{\frac{1}{n+1}}}{u}\right)$;

therefore coefficient of h^{n+1} in u

$$= \text{coefficient of } \frac{1}{u} \text{ in } \frac{X}{(n+1).u^{n+1}} = \frac{d^n X}{1.2...\overline{n+1}}.du^n$$

when u is put $= 0$ by Maclaurin.

Put the same equation under the form

$$z = h.\frac{z}{u}.X^{\frac{1}{n+1}},$$

then by Section (2)

$$u \text{ or } hX^{\frac{1}{n+1}} = \text{coefficient of } \frac{1}{z} \text{ in } -h\frac{dX^{\frac{1}{n+1}}}{dz}\,\text{l.}\left(1 - \frac{h}{z}.\frac{z}{u}X^{\frac{1}{n+1}}\right);$$

$\therefore$ coefficient of h^{n+1} in u

$$= \text{coefficient of } \frac{1}{z} \text{ in } \frac{1}{z^n} \cdot \frac{X^{\frac{n}{n+1}}}{n} \cdot \frac{dX^{\frac{1}{n+1}}}{dz} \cdot \left(\frac{z}{u}\right)^n$$

$$= \text{coefficient of } z^{n-1} \text{ in } \frac{1}{n \cdot \overline{n+1}} \cdot \frac{dX}{dz} \cdot \left(\frac{z}{u}\right)^n$$

$$= \frac{d^{n-1}\left\{\frac{dX}{dz} \cdot \left(\frac{z}{u}\right)^n\right\}}{1.2 \ldots \overline{n+1} dz^{n-1}}.$$

Equate this with the former value and we get the above theorem.

To find the sum of any function of the m least roots of the equation

$$(z - a)^m = h \cdot F(z)$$

as $f(a_1) + f(a_2) + \ldots\ldots f(a_m)$.

Put

$$z = x + a; \quad \therefore \quad x^m = hF(a + x)$$

to find

$$f(a + \beta_1) + f(a + \beta_2) + \&c.$$

β_1, β_2, &c. being the m least values of x, we have, by Section 3, the required sum

$$= mf(a) - \text{coefficient of } \frac{1}{x} \text{ in } f'(a + x) \, l. \left\{1 - \frac{h}{x^m} \cdot F(a + x)\right\}:$$

whence, if we expand the log. this becomes

$$= mf(a) + h \cdot \frac{d^{m-1}\{f'(a) \cdot F(a)\}}{1.2 \ldots \overline{m-1} \cdot da^{m-1}} + \frac{h^2}{2} \cdot \frac{d^{2m-1}\{f'(a) \cdot \overline{F(a)}^2\}}{1.2 \ldots \overline{2m-1} \cdot da^{2m-1}} + \&c.$$

In this remarkable theorem if we put $m = 1$ it becomes Lagrange's; but if we put $m =$ the dimensions of the equation, then the m least are in reality all the roots, so that this case corresponds to the theorem given by M. Cauchy in Vol. IX. Memoirs of the Institute.

Let us retake for a moment the same equation

$$(z - a)^m = h F(z),$$

if we extract the mth root of both sides and suppose 1, ρ, $\rho^2 \ldots \rho^{m-1}$ to be the mth roots of unity, we may form m different equations, viz.

$$z = a + h^{\frac{1}{m}} . \overline{F(z)}^{\frac{1}{m}}$$

$$z = a + \rho . h^{\frac{1}{m}} . \overline{F(z)}^{\frac{1}{m}}$$

$$z = a + \rho^2 . h^{\frac{1}{m}} . \overline{F(z)}^{\frac{1}{m}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$z = a + \rho^{m-1} . h^{\frac{1}{m}} . \overline{F(z)}^{\frac{1}{m}}$$

to each of these equations apply Lagrange's Theorem to determine $f(z)$, we thus get m values, viz.

$$f(z) = f(a) + h^{\frac{1}{m}} . f'(a) . \overline{F(a)}^{\frac{1}{m}} + \frac{h^{\frac{2}{m}}}{1 . 2} . \{f'(a) . \overline{F a^{\frac{2}{m}}}\}' + \&c.$$

$$f(z) = f(a) + \rho h^{\frac{1}{m}} f'(a) . \overline{F(a)}^{\frac{1}{m}} + \frac{\rho^2 h^{\frac{2}{m}}}{1 . 2} . \{f'(a) . \overline{F(a)}^{\frac{2}{m}}\}' + \&c.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f(z) = f(a) + \rho^{m-1} . h^{\frac{1}{m}} . f'(a) . \overline{F(a)}^{\frac{1}{m}} + \frac{\rho^{2 . \overline{m-1}} . h^{\frac{2}{m}}}{1 . 2} \{f'(a) \overline{F(a)}^{\frac{2}{m}}\}' + \&c.$$

Adding up all these equations, and recollecting by the properties of the roots of unity that their sum, or the sum of any of their powers is 0, except in the case of the mth, $2m^{th}$, &c. powers, when the sum is m, we evidently get for the sum of the function

of the m roots precisely the same result as before obtained in p. 141.

We see likewise from this, that if there is an equation of n dimensions, and we transpose the highest power, so as to put the equation under the form $x = \sqrt[n]{P}$, then applying Lagrange's Theorem, to find x, and multiplying the first group of n terms by

$$1,\ \cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n} + \sqrt{-1}\sin\frac{4\pi}{n},\ \&c.$$

and the 2nd, 3rd, &c. groups of n terms respectively by the same, we shall get a 2nd root of the equation, and multiplying them by

$$1,\ \cos\frac{4\pi}{n} + \sqrt{-1}\sin\frac{4\pi}{n},\ \&c.$$

we shall get a 3rd root, and so on for all the roots.

Suppose a series is to be reverted, or an equation solved, such as $a_0 = x - a_1x^2 - a_2x^3$, &c., we can easily express the law of the general term in the value of x, thus by Section I.

$$x = \text{coefficient of } \frac{1}{x} \text{ in } -1.\left\{1 - \left(\frac{a_0}{x} + a_1x + a_2x^2,\ \&c.\right)\right\};$$

therefore the general term of x = coefficient of $\frac{1}{x}$ in

$$\frac{1}{n}\left(\frac{a_0}{x} + a_1x + a_2x^2,\ \&c.\right)^n.$$

Now $\dfrac{\left(\frac{a_0}{x} + a_1x + a_2x^2,\ \&c.\right)^n}{1.2\ldots n}$ = coefficient of α^n in $\epsilon^{\frac{\alpha a_0}{x}}.\epsilon^{\alpha a_1 x}.\epsilon^{\alpha a_2 x^2}$, &c.

Hence the general term of x = coefficient of $1.2\ldots\overline{n-1}.\frac{\alpha^n}{x}$ in the following product, viz.

$$\left(1 + \alpha\frac{a_0}{x} + \frac{\alpha^2a_0^2}{1.2.x^2} + \&c.\right)\left(1 + \alpha a_1x + \frac{\alpha^2a_1^2x^2}{1.2},\ \&c.\right)(1 + \alpha a_2x^2 + \&c.),$$

i. e. it $=\Sigma \frac{1.2 \ldots \overline{n-1} . a_0^{b_0} . a_1^{b_1}, \text{ \&c.}}{1.2 \ldots b_0 \times 1.2 \ldots b_1, \text{ \&c.}}$ subject to the two conditions

$$b_0 + b_1 + b_2, \text{ \&c.} = n,$$

$$b_0 - b_1 - 2b_2, \text{ \&c.} = 1.$$

When to an equation $F(x)=0$ we add a very small term $e.\phi(x)$, what is the corresponding correction to be applied to the root? The root of $F(x)=0$,

$$\text{is the coefficient of } \frac{1}{x} \text{ in } -1. \frac{F(x)}{x},$$

and the root of $F(x)+e\phi(x)=0$

$$\text{is the coefficient of } \frac{1}{x} \text{ in } -1. \left\{\frac{F(x)}{x} + \frac{e\phi x}{x}\right\};$$

therefore the increment of the root

$$= \text{coefficient of } \frac{1}{x} \text{ in } -1. \left(\frac{Fx}{x} + \frac{e\phi x}{x}\right) + 1. \frac{Fx}{x}$$

$$= \text{coefficient of } \frac{1}{x} \text{ in } 1. \left\{1 + \frac{e\phi(x)}{F(x)}\right\}$$

$$= \text{coefficient of } \frac{1}{x} \text{ in } \frac{e\phi(x)}{F(x)}$$

when $e\phi(x)$ is supposed to be very small.

COR. When the proposed equation is $z = a + f(z)$, which by adding an additional small term, becomes

$$z = a + f(z) + e\phi(z),$$

the correction to be applied to the root is

$$e\left\{\phi(a) + [\phi(a)\overline{fa}^2]' + \frac{[\phi(a)\overline{fa}^3]''}{1.2} + \text{\&c.}\right\}.$$

To find the value of the error committed when we stop at the nth term of Lagrange's series.

If we pursue the same method as that used at the beginning of this Section, we get the required error

$$= \text{coefficient of } \frac{1}{x} \text{ in } \frac{h^{n+1}}{n+1 \cdot x^{n+1}} f'(a+x) \cdot F(a+x)^{n+1}$$

$$+ \frac{h^{n+2}}{x+2 \cdot x^{n+2}} \cdot f'(a+x) \cdot F\overline{a+x}^{n+2}, \text{ \&c.}$$

$$= \text{coefficient of } \frac{1}{x} \text{ in } f'(a+x) \left\{ \left(\frac{h \cdot F(a+x)}{x} \right)^{n+1} \cdot \frac{1}{n+1} \right.$$

$$\left. + \left(\frac{h \cdot F(a+x)}{x} \right)^{n+2} \cdot \frac{1}{n+2} + \text{\&c.} \right\}.$$

$$= \text{coefficient of } \frac{1}{x} \text{ in } f'(a+x) \cdot \int_h \left\{ \left(\frac{F(a+x)}{x} \right)^{n+1} \cdot h^n \right.$$

$$\left. + \left(\frac{F(a+x)}{x} \right)^{n+2} \cdot h^{n+1} + \text{\&c.} \right\}.$$

the integral commencing from $h=0$.

But the part under the sign of integration may be summed, and

$$= \frac{\left(\frac{F(a+x)}{x} \right)^{n+1} \cdot h^n}{1 - h \cdot \frac{F(a+x)}{x}}$$

$$= \frac{\overline{F(a+x)}^{n+1}}{x^n} \cdot \frac{h^n}{x - h F(a+x)};$$

therefore the error = coefficient of x^{n-1} in

$$f'(a+x) \cdot F(a+x)^{n+1} \cdot \int_h \frac{h^n}{x - h F(a+x)},$$

to be calculated by Definite Integrals, in the next Section.

SECTION V.

LET $F(x)$ be any function of x, containing only integer powers of x, positive, or negative; as

$$F(x)=A+Bx+Cx^2+\&c.$$

$$+\frac{b}{x}+\frac{c}{x^2}+\&c.$$

Put ϵ^θ for x, multiply then by ϵ^θ and integrate with respect to θ, we thus obtain

$$\int_\theta F(\epsilon^\theta).\epsilon^\theta=\text{const.}+A\epsilon^\theta+\frac{B\epsilon^{2\theta}}{2}+\frac{C\epsilon^{3\theta}}{3}+\&c.$$

$$+b.\theta-c\epsilon^{-\theta}-\&c.$$

All the terms in this result, except $b.\theta$ and the const., are circulating or periodical terms; i. e. if we give θ the series of imaginary values comprised in the formula $\theta+\alpha\sqrt{-1}$, then for any two values of α, which differ by (2π) a whole circumference, these terms have precisely the same value, but the term $b.\theta$ becomes $2\pi b\sqrt{-1}$ between the above limits: thus it appears that the coefficient of the first negative power of x in

$$F(x)=\frac{1}{2\pi\sqrt{-1}}.\int_\theta F(\epsilon^\theta).\epsilon^\theta;$$

the integral being taken through one entire circulation, i. e. from θ to $\theta+2\pi\sqrt{-1}$.

Restore now for ϵ^θ its value x; then $\int_\theta F(\epsilon^\theta).\epsilon^\theta$ becomes $\int_x F(x)$ which by actual integration is

$$\{\text{const.} + A.x + B.\frac{x^2}{2} + \&c.$$

$$+ b.\text{l.}(x) - \frac{c}{x} - \&c.\}.$$

And the limiting values of x, which correspond to the former limits of θ are ϵ^θ, and $\epsilon^{\theta+\sqrt{2\pi-1}}$, both of which are equal. But the integral does not vanish, though the numerical values of the limits are equal; for it has been above shewn to be equal to $2\pi b\sqrt{-1}$, which may be easily explained; thus,

$$Ax \text{ or } A\epsilon^\theta \text{ between limits} = A\epsilon^{\theta+2\pi\sqrt{-1}} - A\epsilon^\theta$$

$$= A\epsilon^\theta.(\epsilon^{2\pi\sqrt{-1}} - 1) = 0.$$

$$\text{Similarly } \frac{Bx^2}{2} \text{ between limits} = 0, \&c.$$

$$\text{but } b\,\text{l.}\,x \text{ between limits} = b\,\text{l.}\,\epsilon^{\theta+2\pi\sqrt{-1}} - b\,\text{l.}\,\epsilon^\theta$$

$$= b(\theta + 2\pi\sqrt{-1}) - b\theta$$

$$= 2\pi b\sqrt{-1},$$

though, therefore, the quantity under the transcendent sign of log. is the same, at both the limits of the integral; yet, if we suppose x to circulate through a series of values, until it again arrives at the value from which it set out, we must then make use of the multiplicity of the values of the log. (x), (which are all included under the form $2m\pi\sqrt{-1}$ + real log. x) by giving to it, at its limit, that value it acquires by one circulation, namely $2\pi\sqrt{-1}$ + real log. (x).

It is the same, with other transcendent functions arising from integration, and possessing the character of having an infinity of values in arithmetical progression; as $\sin^{-1}(x)$, $\tan^{-1}(x)$, &c.:

the common difference of that progression is the quantity by which they alter through one circulation; which is therefore

2π for $\sin^{-1}(x)$; π for $\tan^{-1}x$, and $2\pi\sqrt{-1}$ for $\log(x)$:

if then at the limits of an integral, the quantity under any of these signs should be the same; and if x has been supposed to have passed through one circulation, from its leaving a certain limit, until it has returned to the same, then for the corresponding part of the definite integral, we evidently must not put 0, but the common difference of the above-named progression.

Thus, in finding the area of a circle $\int\frac{dx}{\sqrt{1-x^2}}$, taken from $x=1$ until $x=1$ after one circulation, the value is $\sin^{-1}(1)-\sin^{-1}(1)$, but the former must be manifestly understood to exceed the latter by 2π.

If this distinction be attended to, there will arise no difficulty, in calculating the values of terms of this nature when they enter definite integrals, as they frequently do; we have also here the advantage of seeing the analytical use of the multiplicity of the values of these kind of functions.

It has been proved, (p. 146), that the coefficient of the first negative power of x in $F(x)$ is equal to $\int_\theta F(e^\theta).e^\theta$, taken from θ to $\theta+2\pi\sqrt{-1}$ or through one circulation. Applying this to the theorems given in the former Sections, we get the following results.

If $\phi(x)=0$ be any equation containing only positive and integer powers of x, then

$$2\pi\sqrt{-1}\times \text{ the least root } = -\int_\theta e^\theta . \text{l.}\{\phi(e^\theta).e^{-\theta}\}$$

$$2\pi\sqrt{-1}\times \text{ the sum of } m \text{ least roots } = -\int_\theta e^\theta \text{ l.}\{\phi(e^\theta).e^{-m\theta}\}.$$

Similarly, $2\pi\sqrt{-1}\times$ any function $f(\alpha)$ of the least root is

$$= 2\pi\sqrt{-1}f(0) - \int_\theta \frac{d.f(\epsilon^\theta)}{d\theta}.\,\text{l.}\,\{\phi(\epsilon^\theta).\epsilon^{-\theta}\}$$

and $2\pi\sqrt{-1}\times$ the sum of that function of the m least roots is

$$2m\pi\sqrt{-1}f(0) - \int_\theta \frac{d.f(\epsilon^\theta)}{d\theta}.\,\text{l.}\,\{\phi(\epsilon^\theta).\epsilon^{-m\theta}\}$$

the integrals, in all these cases, being taken from θ to $\theta+2\pi\sqrt{-1}$, or through one circulation of x. If the functions $f(x)$ and $\phi(x)$ contain negative or fractional powers of x, we must put $x=a+z$, and proceed with z as the unknown quantity.

In like manner the error made by stopping at the nth term of Lagrange's series given in p. 145.

$$= \int_\theta \int_h \frac{h^n f'(a+\epsilon^\theta).\overline{F(a+\epsilon^\theta)}^{n+1}}{\epsilon^{n\theta} - h\,\epsilon^{n-1.\theta}F(a+\epsilon^\theta)}$$

the integral with respect to h commencing when $h=0$.

With one application we shall terminate this Section.

Let $x^2+ax+b=0$, to find x.

$$2\pi\sqrt{-1}\times \text{ root } = -\int_\theta \epsilon^\theta\,\text{l.}\,(\epsilon^\theta + a + b\epsilon^{-\theta})$$

$$= -\,\text{l.}\,(\epsilon^\theta + a + b\epsilon^{-\theta}).\int_\theta \epsilon^\theta + \int_\theta \frac{\epsilon^{2\theta}-b}{\epsilon^\theta + a + b\epsilon^{-\theta}}.$$

Now $\int_\theta.\epsilon^\theta$ evidently vanishes between limits;

$$\therefore\ 2\pi\sqrt{-1}\times \text{ root} = \int_\theta \frac{\epsilon^{2\theta}-b.\epsilon^\theta}{\epsilon^{2\theta}+a\epsilon^\theta+b}$$

$$= \int_\theta \epsilon^\theta - \int_\theta \frac{a\epsilon^{2\theta}+2b\epsilon^\theta}{\epsilon^{2\theta}+a\epsilon^\theta+b}$$

$$= -\frac{a}{2}\int_\theta \frac{d(\epsilon^{2\theta}+a\epsilon^\theta+b)}{\epsilon^{2\theta}+a\epsilon^\theta+b} + \left(\frac{a^2}{2}-2b\right)\int \frac{d.\epsilon^\theta}{\left(\epsilon^\theta+\frac{a}{2}\right)^2 + \left(b-\frac{a^2}{4}\right)},$$

$$\text{or } 2\pi\sqrt{-1}\,.\,\text{root}$$

$$= -\frac{a}{2}\,\text{l.}\,(\epsilon^{2\theta} + a\epsilon^{\theta} + b) + \frac{\frac{a^2}{2} - 2b}{\sqrt{b - \frac{a^2}{4}}}\tan^{-1}\frac{\epsilon^{\theta} + \frac{a}{2}}{\sqrt{b - \frac{a^2}{4}}} + \text{constant},$$

and taking these transcendent functions (which enter directly by integration) from θ to $\theta + 2\pi\sqrt{-1}$, the expressions under them do not alter at these limits, and therefore, by the preceding remarks, the value of

$$\text{the log. between limits is } 2\pi\sqrt{-1}$$

$$\text{of } \tan^{-1}\ldots\ldots\ldots\ldots\ldots\ldots \quad \pi;$$

$$\therefore\ 2\pi\sqrt{-1}\ .\ \text{root} = -2\pi\sqrt{-1}\,.\frac{a}{2} + \pi\,.\,\frac{\frac{a^2}{2} - 2b}{\sqrt{b - \frac{a^2}{4}}},$$

$$\text{or root} = -\frac{a}{2} + \sqrt{\frac{a^2}{4} - b}\,.$$

It need scarcely be added, that this instance is only brought as an illustration of the operation.

The method, here adopted, for finding the coefficient of the first negative power of x, in any function of x; on which the value of the root of an equation has been made to depend, was chosen, as leading to the preceding remarks on Definite Integrals: the application of either Parseval's Method, or of a theorem given in my former paper, *on Definite Integrals;* (Vid. *Cambridge Transactions*, Vol. III. p. 437) to the theorems established in the first 4 Sections, would have sufficed to give other analytical expressions, for the roots of equations, &c. also, in Definite Integrals.

SECTION VI.

We shall here add a few theorems, connected with the present subject.

To transform the definite integral of f (x), ***taken between two limits, which are the two roots of given equations; to another, in which the limits are known quantities.***

Suppose $\phi(x) = 0$, $F(x) = 0$ are the two equations, the roots of which, are to be the limits of the $\int f(x)$; and first, let us consider the value of this integral, taken from $x = 0$, to $x =$ the root of the equation $\phi(x) = 0$; this value is a certain function of that root; and is therefore, by Section (2), = coefficient of $\frac{1}{x}$, in $-1.\frac{\phi(x)}{x}$ multiplied by the derived function, namely, by $f(x)$; and applying the theorems given in Section 5, it follows that, $\int f(x)$ is

$$= -\int_\theta \frac{\epsilon^\theta . f(\epsilon^\theta) . 1. \{\phi(\epsilon^\theta) . \epsilon^{-\theta}\}}{2\pi\sqrt{-1}}$$

the limits being 0 and $2\pi\sqrt{-1}$.

Similarly, $\int f(x)$ from $x = 0$ to $x =$ root of $F(x) = 0$

$$= -\int_\theta \frac{\epsilon^\theta f(\epsilon^\theta) . 1. \{F(\epsilon^\theta) . \epsilon^{-\theta}\}}{2\pi\sqrt{-1}}$$

between same limits.

Hence, if we subtract, we get $\int f(x)$ between the two proposed limits, viz.

$$\int_\theta \frac{\epsilon^\theta . f(\epsilon^\theta)}{2\pi\sqrt{-1}} . 1. \frac{F(\epsilon^\theta)}{\phi(\epsilon^\theta)} \qquad \begin{cases} \theta = 0 \\ \theta = 2\pi\sqrt{-1} \end{cases}$$

the forms of the functions being such as those already noticed.

In many instances we have had occasion to take the coefficient of $\frac{1}{x}$ in $f(x)$, or which is the same, the term independent of x in $x f(x)$, there may easily be obtained several theorems, for facilitating the research of this quantity, the following may perhaps, from the elegance of its form, be not deemed unworthy of notice, viz.

The term independent of x in $\phi\left(x - \frac{1}{x}\right)$

$$= \phi(x) - \{x\phi'(x)\}' + \frac{\{x^2\phi''(x)\}''}{1^2 . 2^2} - \frac{\{x^3\phi'''(x)\}'''}{1^2 . 2^2 . 3^2} + \&c.,$$

x being put $= 1$ in the series.

For $\phi(x)$ may be represented by $\Sigma a_n . x^n$, the symbol Σ, denoting *the sum* with respect to n,

$$\text{Hence } \phi\left(x - \frac{1}{x}\right) = \Sigma a_n . \left(x - \frac{1}{x}\right)^n.$$

$$\text{But } \left(x - \frac{1}{x}\right)^n = \left(1 - \frac{1}{x}\right)^n . (1 + x)^n;$$

the part of which, independent of x, is

$$= 1 - n^2 + \left(\frac{n . \overline{n-1}}{1 . 2}\right)^2 - \&c.$$

$$\text{or} = x^n - \left(x . \frac{d . x^n}{dx}\right)' + \frac{\left(x^2 . \frac{d^2 . x^n}{dx^2}\right)''}{1^2 . 2^2}, \&c.$$

supposing x to be put $= 1$, after the differentiations are performed.

Multiply by a_n, and take the Σ with respect to n; observing, generally, that

$$\Sigma a_n d^m \frac{\left(x^m . \frac{d^m . x^n}{dx^m}\right)}{dx^m} = \frac{d^m . \left(x^m \frac{d^m . \phi(x)}{dx^m}\right)}{dx^m},$$

and we obtain the proposed theorem.

In concluding these few observations on Algebra, we may observe, that in whatever department of Analysis we seek principles, simple in their announcement, and general in their applications; so far as we succeed, something useful is acquired for analysis, nor can any branch of Mathematics, however humble, be deemed unworthy of cultivation in this way, and it is with this view that the Author has presented this Essay to the Society.

ROBERT MURPHY.

CAIUS COLLEGE,
March, 1831.

ERRATA.

P. 140. l. 18. put du^n below the line.

— 144. bottom, *for* $\overline{f(a)^2}$ *put* $f(a)$.

and *for* $\overline{f(a)^3}$ *put* $\overline{f(a)}^2$.

— 147. l. 5. *for* $\sqrt{2\pi-1}$ *put* $2\pi\sqrt{-1}$.

V. *Mathematical Exposition of some of the Leading Doctrines in Mr. Ricardo's "Principles of Political Economy and Taxation."*

By the Rev. W. WHEWELL, A.M.

FELLOW AND TUTOR OF TRINITY COLLEGE.

[Read *April* 18, and *May* 2, 1831.]

1. Among a number of those who have recently cultivated the study of Political Economy, an opinion appears to prevail that, by the labours of Mr. Ricardo and his followers, a large mass of our knowledge on this subject has been reduced to the form of a series of exact logical deductions from a few simple and evident principles. If this were really the case—if the fundamental principles were completely enumerated and clearly established, and if their consequences were truly and fully traced—this branch of knowledge might rightly be considered as having assumed an exact and scientific character. My present purpose is not to examine how far these claims are well founded; but to shew in what manner those portions of the speculations here referred to, which pretend to such a scientific form, may be advantageously submitted to mathematical investigation. I have already, in a former communication, observed that when our object is to deduce the results of a few precise and universal principles, mathematical processes offer to us both the readiest and safest method; since by them we can most easily overcome all the

difficulties and perplexities which may occur in consequence of any complexity in the line of deduction, and are secure from any risk of vitiating the course of our reasoning by tacit assumptions or unsteady applications of our original principles. Perhaps it might not be difficult to shew that in several speculations on these subjects, such errors have not been altogether avoided. And there is probably a considerable class of readers who will find the doctrines of Political Economy, when put in a mathematical shape, more clear, compendious, and manageable than in the works to which I refer. I may add also, that the mathematical formulæ which I shall obtain, will be, at the same time, both much more exact and much more general than the numerical examples in which writers on the subject are in the habit of embodying and illustrating their reasonings.

2. I have said that the doctrines of Political Economy have been supposed to be reduced to a few simple fundamental principles; and my business will be to state these principles and to trace their consequences. The mathematical investigation proceeds from these principles as *postulates*, and has no concern with their truth or falsehood. To extricate such principles from the mass of facts which observation of the world and of ourselves teaches us,—to establish the reality, number, and limits of such laws,—these are offices which belong to a branch of philosophy altogether different from that with which the mathematician, in the proper sense of the name, has to deal. Such a task is far from being either short or easy; and it would be very hasty to take for granted that it is already completed, or even that any considerable portion of it is performed. For my own part, I do not conceive that we are at all justified in asserting the principles which form the basis of Mr. Ricardo's system, either to be

steady and universal in their operation, or to be of such paramount and predominant influence, that other principles, which oppose and control them, may be neglected in comparison. Some of them appear to be absolutely false in general, and others to be inapplicable in almost all particular cases. Perhaps, however, to trace their consequences may be one of the most obvious modes of verifying or correcting them.

I proceed now to enumerate the Postulates which seem to form the foundations of Mr. Ricardo's doctrines: and I shall point out at the same time the manner in which they may be mathematically expressed and treated.

I. *Postulate of Rent.*

3. In the system now before us, agriculture is considered as an employment of capital; and the farmer as a person who lives on his profits, and who can and will remove his capital to another employment, if his profits can be so increased, or so prevented from falling. Hence the farmer will not consent to make less than the average rate of profits; also, competition will not allow him to make more: and therefore the excess of the produce of the land above that amount which is necessary to realize such profits, will be transferred to the landlord as *rent*. Hence, on these suppositions, rent is the excess of the produce of capital employed on land, above the produce of the same capital otherwise employed.

It is also supposed that there are soils of different degrees of fertility, which form a continuous decreasing series: and that the lowest degree of fertility on which the cultivator can obtain a living profit without paying rent, will be cultivated.

It is also supposed that there are modes of employing, on the same land, successive quantities on *doses* of capital, and that

each successive dose must universally obtain a proportionally smaller return than the preceding.

Of these hypotheses, the first is generally allowed to be approximately verified in England: though even here, the reluctance, difficulty and loss which accompany the transfer of farming capital to other employments, the moral and social ties which connect the landlord and tenant, and the numbers of cases in which the cultivator does not live on profits only, very much limit the generality, and obstruct and extend the manifestation of the naked principle thus asserted. And, taking the world at large, it has been shewn, in the admirable work of Mr. Jones, that this view and measure of rents is entirely inapplicable, and that none of the suppositions on which it proceeds have the slightest resemblance to the actual state of things. It is to be recollected therefore, that whenever the postulate of rent is introduced, the application of our reasonings can only be made to cases of *farmer's rents*: and that all countries in which *serf*, *metayer*, *ryot* or *cottier* rents, or *cultivator-proprietors* prevail, that is 99 hundredths of the cultivated globe, are to be excluded from our conclusions.

The existence of a limiting soil, or of a soil cultivated but paying no rent, is not necessarily implied in the measure of rent just mentioned. The fact that there are soils approaching nearly to this limit may however be conceded, and will not, in most cases, much affect the conclusion. On this supposition, rent is the excess of the produce of the soil over that of an equal portion of the limiting soil.

The third supposition, that each successive *dose* of capital must procure a proportionally smaller return, appears to be, as

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* Essay on the Distribution of Wealth, and the Sources of Taxation. Murray, 1831.

Mr. Jones has observed, an assumption without any *a priori* foundation whatever, and not at all consistent with the known history of agriculture.

Let r be the number of *quarters of corn*, grown on an *acre* of land, p the price of a quarter in *pounds*, c the capital employed on *one acre*, including wages; d the sum requisite to replace c with the usual profit: then we have the money rent of *one acre* $= pr - d$.

If r_n be the produce of the limiting soil, $pr_n - d = 0$; and

$$\text{rent of } \textit{one acre} = pr - pr_n.$$

Since c is the whole capital employed on an *acre*, it is the sum of all the *doses* up to the last.

Instead of a *quarter of corn*, a *pound*, *an acre*, we may take any other unit of produce, price, and land.

II. *Postulate of Wages.*

4. Mr. Ricardo assumes that the natural rate of wages is invariable; that is, that the labourer's command of food and other necessaries is never permanently augmented or diminished. Hence, if the price of corn (or whatever is the main article of food) rises or falls, a rise or fall in wages shortly follows and compensates this charge.

This opinion is supposed to be established by the ascertained laws of the progress of population. It is conceived that if the demand for labour, and consequently the reward of it, is diminished, the encouragement to population being thus weakened a retardation in its advance will occur, which will, after a certain period, restore the original standard of wages: and that these effects inverted will occur in the case of an increase of wages.

The truth of such a proposition is repeatedly taken for granted by Mr. Ricardo (3d ed. p. 88. 95. 118. 173, &c.), and is employed by him as a leading step in several of his reasonings. It would seem however that the assumption of such a necessary and universal operation of wages upon population, is entirely gratuitous and unfounded, and that we have not the slightest ground for asserting this to be the law of such changes, any more than any other law arbitrarily assumed.

We might for instance assume that a rise of wages, by improving the chance of the labourer to make some saving, will increase his prudence and self-control, and thus diminish the rate at which population increases. Without asserting this to be more certainly true than the opposite postulate, that a rise of wages accelerates the rate of increase, we may with reason maintain it to be as certainly true in some cases: and this is quite enough to shew the baselessness of Mr. Ricardo's postulate as a general law.

If the labourers of any country were placed at the bare limit of possible subsistence, and under a complete privation of prudential restraint;—so that nothing could be taken away from their means without destroying them by want, nor any thing added, without being speedily absorbed by an increase of their numbers;—on such a supposition their condition might be considered as fixed; and the postulate might be conceded. And some such supposition, suggested by erroneous views of the laws of population, appears to have led to the assumption before us.

This supposition however does not, it is to be hoped, represent the condition of any country. It certainly bears no resemblance to that of our own. If we compare the English labourer with the labourer of France, Ireland, or India, we readily per-

ceive how far he is above the point we have described, to which indeed none even of the others can be said to approximate. To consider the English labourer as possessed of mere necessaries, in the severest sense of the word, would therefore be an obvious and glaring falsity. Hence Mr. Ricardo found it convenient to take, not the necessary, but the ***habitual*** subsistence of the labourer as the permanent standard of wages (p. 91. 95). The assumption however that the ***habits*** of the labourers cannot change from period to period, so as to accommodate themselves to different amounts of real wages, appears to be quite unsupported by reason, and in direct contradiction of all known history. The habitual necessaries and comforts of the labourer may, and do, undergo changes simultaneous and co-ordinate with those of the population. Any attempt therefore to derive the fluctuations of the latter, by supposing the former invariable, is utterly visionary and unphilosophical.

To obtain such principles as may truly indicate to us the manner in which alterations of wages do really operate upon the habits and numbers of the labourers, is an object of great interest and importance, and one in which, though some progress has been made, much is yet to be done. It seems little likely that any one general law, free from all control of time, place and circumstance, will be found of any real use or value.

We may indeed notice, in order to avoid, the propensity of the speculative powers of the human mind to rush forwards and to endeavour to seize on such a general law. The professedly hypothetical statement of a geometrical progression of population, and an arithmetical progression of subsistence, made by Mr. Malthus for the purpose of introducing his views, has been far more frequently quoted, than many of the most valuable views of that

eminent philosopher. And if it were to be maintained, for instance, that the rate of increase varies inversely as the existing population of a given district, or if any other mathematical law were asserted; though the probabilities against it would be so strong, as only to be surmountable by an enormous mass of evidence; the law itself might easily be made to attract notice with the pretensions of a discovery.

The following is the manner in which the Postulate now under notice is introduced into calculation.

Let w be the wages of a labourer for *one year*, when the price of *corn* is p. Let a fraction f of the labourer's wages be expended in *corn* (or other necessary food), and the remaining fraction $1-f$ in other commodities which do not rise in price with corn. Now let the price of corn become $p(1+x)$. Then the expenditure of the labourer will be $(1+x)fw$ for corn, and $(1-f)w$, as before, for other things. Therefore the whole wages are now

$$=(1+x)fw+(1-f)w=(1+fx)w.$$

III. *Postulate of Price.*

5. The exchangeable value of a commodity depends solely upon the relative quantity of labour necessary for its production. Or, in other words, articles which require for their production equal quantities of labour will exchange for each other.

It is not difficult to shew that all exchangeable value has its *source* in human labour; and that this is true of raw produce no less than of manufactured goods. Mr. Ricardo's principle goes further, and asserts that the exchangeable value is in proportion to the *quantity* of the labour necessarily employed, without any regard to the wages which have been paid; the capital by which the labour is aided being however estimated as part of

the requisite labour; and the profits upon this capital being allowed in the usual manner.

This measure of exchangeable value is to be understood as applicable only in the long run, and as the regulating and limiting principle of what may be called the *natural price*. Under this restriction its truth depends on the consideration, that if this were not the law of exchange, some labour would be better rewarded than some other, and hence the equilibrium would be restored by a transfer of labourers to the favoured employment. The principle therefore is true so far as the possibility and operation of such a transfer extends; that is, it is true of prices in the same country, and so far as the labour which is embodied in the commodity is, from conditions of time and place, within the reach of this competition.

That some kinds of labour are more highly paid than others does not vitiate this principle; for they are so in virtue of the education required, the skill of the labourer, or the intensity of the labour (*Ric.* p. 11.). And the tendencies which distribute the numbers and rewards of labourers in different employments, may be supposed to have reached their equilibrium; so that a change in the quantity of labour required can alone affect the price.

The principle thus enuntiated is sometimes expressed by saying, that the price must be such as to pay the cost of production with profits.

This *natural price* may often be different from the *market price*, which is affected by another principle, that of *supply* and *demand*. According to the principle of supply and demand, the price increases by an increase of the demand, or by a diminution of the supply, and *vice versa*. The exact law which connects these changes of price with those of supply and demand,

has not been ascertained with any precision, and is probably very different in different cases. When it is necessary to introduce it into the calculation, I shall suppose (as in a former Memoir) that the increase of price is proportionally greater than the diminution of supply.

The principle of supply and demand effects the *market price* immediately and at once. If the market price rise, the supply is increased, and the price thus brought down, and in this manner the market price gravitates to the natural or remunerating price, which is the standard in the long run.

Let a capital c produce r units of a commodity, p being the price of one unit. Let b be the return which is annually necessary to replace the circulating part of the capital c and the wear and tear of the fixed part: and let γ be the rate of profit. Then $pr=b+\gamma c$, is the equation which determines p.

Also if the supply diminish in the ratio $1 : 1-y$, the price increases in the ratio $1 : 1+ey$: (e being greater than 1).

IV. *Postulate of Profit.*

6. Profits are entirely regulated by Wages.

Wages and Price being determined by the two preceding postulates, and the power of production of a given capital being given, the profit left for the capitalist is necessarily determined, being that portion of the surplus return which is not absorbed by wages. In this manner profits appear to be an entirely passive portion of the produce: and when they fall, the capitalist has no remedy in any possibility of raising prices, or otherwise compensating his loss.

This is true so long as we suppose capital confined to one country. When, however, profits fall below a certain point, the

capitalist will either cease to retain his capital in its office, and will convert a portion of it into expenditure: or he will transfer it to some other country where profits are higher. This resource is one which, though probably not called into action by a difference of one or two per cent. between domestic and foreign profits, will undoubtedly become very active and influential, if the difference should go much further, and will operate by diminishing the supply of the capital at home, till profits become of such a magnitude as to be a sufficient inducement to the employment of capital at home.

I shall suppose γ to represent the profit on £1; so that if profits be 10 per cent., γ is $\frac{1}{10}$, or=, 1; if 15 per cent., γ=,15; if 30 per cent., γ=,3. The rate of profit will be 100γ per cent.

V. *Postulate of Equilibrium.*

7. In the reasonings which we have to follow, the principles which we have stated are supposed to be carried into complete operation, and commodities to be distributed according to the laws to which those principles give rise. Thus the rate of profits of agriculture and of other employments are assumed to be equal, it being supposed that capital is tranferred from one employment to another, till such an equilibrium is produced: though this is a process which manifestly would require a considerable time, and would never be completely performed. In the same manner, assuming the postulate of wages (which as I have said is perfectly gratuitous) it is clear that the changes of population which it supposes would require for their accomplishment a considerable course of years, during which new causes and circumstances might come into action, so as entirely to modify the

result, even if the tendency of the original cause had been rightly stated. In the postulate of price, it has already been noticed, that the market price, determined by the immediate action of demand and supply, may be very different from the natural price, determined, according to the postulate, by the cost of production: this latter price being however that under which the equilibrium obtains, and to which the other perpetually bends.

Supposing the preceding postulates true, the problems in which they are applied are much simplified by assuming such an equilibrium to obtain: but along with this simplification we incur a necessary and perpetual, and, it may be, a very considerable deviation from the circumstances of actual fact. In reality, this equilibrium is never attained: probably in most cases it is never approximated to. There is a constant tendency towards the state of things in which the elements of wealth are in this exact balance, but this is a tendency like that which the waters at the source of a river have to descend towards its mouth. We cannot from such a tendency infer that the whole course of a river is at the same level; and just as little may we flatter ourselves that we have solved the problem of the course and distribution of the current of wealth, when we have combined the laws according to which an exact balance might be produced.

We are to recollect therefore, that even if our principles were exact, deductions from them made according to the method we are now following, would give us only a faint and distant resemblance of the state of things produced by the perpetual struggle and conflict of such principles with variable circumstances. Such deductions however would probably have some resemblance, in the general outline of their results, to the true state of things. They would offer to us a *first approximation*:

and in difficult problems of physics, it is precisely by such a simplification as this, that a first approximation is obtained. Thus in the investigation of the problem of the tides, we have a very complex case of the *motion* of a fluid: but Newton's mode of treating the question was, to consider what would be the form of *equilibrium* of the ocean, acted upon by the forces which produce the tides: and this solution of the problem, though necessarily inexact, was accepted as the best which could easily be obtained. The investigations of Laplace and others who have since treated the problem on its true grounds, as a question of hydrodynamics, have shewn that Newton's solution explains rightly the main features of the phenomenon.

In order however that solutions of this nature may have any value, it is requisite that the principles, of which we estimate the operation, should include *all* the *predominant* causes which really influence the result. We necessarily reject some of the circumstances and tendencies which really exist: but we can do this with propriety, only when the effects of these latter agents are, from their small amount or short duration, inconsiderable modifications only of the general results. The quantities which we neglect must be of an inferior *order* to those which we take into account; otherwise we obtain no approximation at all. We may with some utility make the theory of the *tides* a question of equilibrium, but our labour would be utterly misspent if we should attempt to consider on such principles the theory of *waves*.

It appears to be by no means clear that the irregular fluctuations and transitory currents by which the elements of wealth seek their natural level may be neglected in the investigation of the primary laws of their distribution. It is not difficult to con-

ceive that the inequalities and transfers produced by the temporary and incomplete action of the equalizing causes, may be of equal magnitude and consequence with those ultimate and complete changes by which the general tendency of such causes is manifested. A panic may produce results as wide and as important as a general fall of profits.

Of this source of uncertainty in the value of such results as I shall have to obtain, I cannot pretend to determine the extent. But independently of this consideration, the postulates above stated are so far from being adequate properly to represent the general facts, that we can hardly look for any accordance between the calculated and the observed effects. To consider, for instance, the tendency of mankind to increase their numbers as a universal and inevitable law, and to leave out of consideration the co-ordinate and antagonist tendency by which they endeavour to preserve and increase their comforts, is to insure a total dissimilarity between our theory and the actual state of things.

The postulate of equilibrium is introduced to our calculations by the process of putting the other postulates into the form of *equations*. The values of the quantities involved are by this means determined according to the condition of the equilibrium of our principles above stated.

8. I will now mention the general problems which it appears to have been the object of Mr. Ricardo's work to solve; and I will afterwards proceed to obtain mathematical solutions of them on his principles.

If we suppose the population of a country to go on increasing, and the powers of agriculture to remain stationary, it will, in order that the increased numbers may be provided with subsistence, be

necessary that more and more capital and labour should be applied to the task of raising food. If we suppose these to be employed on new land, more and more land will be perpetually cultivated: and if we make the supposition already mentioned, that the land of the country consists of a progression of soils of decreasing fertility, each new soil cultivated, will yield a less surplus produce to the labour employed upon it.

This increase of population, and consequent extension of agricultural labour to less productive soils, Mr. Ricardo conceived to have been the progress of things in this country: and apparently he conceived it also to be the necessary and universal progress of nations. On this supposition his first main problem was to trace the distribution of the various portions of the produce, as wages, rent, and profits, which takes place in the course of this progress.

That this has been the course of events in England seems to be clearly and demonstrably false. After Mr. Jones' reasonings (Essay, chap. vii. sect. 6.), I do not conceive that any doubt can remain on the subject. And it is remarkable that Mr. Ricardo's error in this instance is not a mistaken assumption of principles, but it is a defect in his deduction from his principles, a part of his task which is generally supposed to be unexceptionable. The error resides in his having neglected altogether the effects of an increase in the power of argriculture, which, in England, has been a change at least as important and as marked, as the increase in the population. This being the case, it is evident that the whole of his assumption of the nature of the economical progress of this country, and the views of the distribution of wealth arising from this assumption, must fall to the ground. It may however still be curious to see the exact consequences of the assumptions now referred to; and moreover, our formulæ

will be applicable with no great modification, to the case in which the powers of production are supposed to have increased.

Mr. Ricardo's other problems are these.

In the course of the progress of things above described, taxes being levied on any portion of the wealth of the country (as wages, rents, land, profits, &c.) to determine their ultimate incidence and consequences.

Mr. Ricardo has also considered some of the questions connected with the subject of foreign trade and the varying value of money, which I shall afterwards endeavour to state.

PROGRESSIVE DISTRIBUTION OF PRODUCE.

9. Let the following notation be employed;

r = the produce on one acre, p = the price of one quarter, c the capital on one acre, γ the rate of profit on £1, l the number of labourers employed on one acre, (including the labour requisite to replace the waste of fixed capital) w the wages of labour.

The value of the return on one acre is pr, of which the labourers' portion is lw, and the capitalist's is γc; hence by Post. I. $pr - lw - \gamma c$ is the rent. On the limiting soil this gives $pr = lw + \gamma c$. Now suppose prices to rise to p', and cultivation to be pushed upon a worse limiting soil, on which the produce is r', the capital c', and the labourers l'. Hence $p'r' = l'w' + \gamma' c'$. And in this case the rent on one acre of the former limiting soil becomes $p'r - lw' - \gamma' c$.

The increase of price is supposed to be in proportion to the increase of labour requisite to produce the same quantity of corn (see Post. III). The whole price pr was, in the first instance, paid for the result of the labour l; the whole price $p'r'$ is afterwards paid for the result of the labour l'; therefore $pr : p'r' :: l : l'$.

On the second limiting soil

$$p'r' = l'w' + \gamma' c'; \text{ whence } \gamma' = \frac{p'r' - l'w'}{c'},$$

$$\text{or since } p'r'l = prl', \quad \gamma' = \frac{(pr - lw')\, l'}{c'l}.$$

$$\text{Hence } \gamma' c = (pr - lw')\frac{cl'}{c'l}.$$

Let the price increase, so that $p' = p(1+x)$; then (Post. II.)

$$w' = w(1+fx).$$

Also let $pr = nlw$. Then we have, for one acre of the first soil, after the second is cultivated,

$$\text{Wages} = lw' = (1+fx)\, lw.$$

$$\text{Profits} = \gamma' c = \{nlw - (1+fx)\, lw\}\frac{cl'}{c'l}$$

$$= (n-1-fx)\, lw\frac{cl'}{c'l}.$$

$$\text{Rent} = p'r - lw' - \gamma' c$$

$$= (1+x)\, nlw - (1+fx)\, lw - \gamma' c$$

$$= \left\{n - 1 + nx - fx - (n-1-fx)\frac{cl'}{c'l}\right\} lw.$$

These are the amounts in money. The amounts of these quantities in corn will be found by dividing each by the price

$$p' = p(1+x).$$

If we suppose the capital on an acre of the new soil to be greater in proportion as the labour employed upon it is less, we have $cl' = c'l$: and in this case, for one acre.

$$\text{Wages} = (1+fx)\, lw.$$

$$\text{Profits} = (n-1-fx)\, lw.$$

$$\text{Rent} = nxlw.$$

The amount of capital and labour which will be employed on an acre of the new soil will depend upon its agricultural constitution. The farmer will fix upon such quantities as give him the most profitable returns.

In the case supposed by Mr. Ricardo, p. 115, $l' = l$, $c' = c$. Hence the last formulæ apply; and we have for the portions of the value of the produce

	Rent.	Profits.	Wages.
at first, price $= p$,	0	$(n-1)lw$	lw
afterwards, price $= p'$,	$nxlw$	$(n-1-fx)lw$	$(1+fx)lw$
fractions of the whole	$\frac{x}{1+x}$	$\frac{n-1-fx}{n(1+x)}$	$\frac{1+fx}{n(1+x)}$

the whole produce is, in the first case, in money pr or nlw, and in corn r: in the second case it is in money $p'r$ or $(1+x)nlw$, and in corn r, as before.

$$\text{We had } \gamma' = \frac{(pr - lw')\,l'}{c'l}: \text{ similarly } \gamma = \frac{pr - lw}{c};$$

$$\text{therefore } \frac{\gamma'}{\gamma} = \frac{pr - lw'}{pr - lw} \cdot \frac{cl'}{c'l} = \frac{n-1-fx}{n-1} \cdot \frac{cl'}{c'l}.$$

In the case taken by Mr. Ricardo, as before, $cl' = c'l$. Also $\gamma = ,16$; $n = 3$; $fx = \frac{1}{10}$;

$$\text{therefore } \gamma' = \frac{19}{20} \times ,16 = ,152.$$

From these formulæ we may obtain the numerical results which Mr. Ricardo has given at p. 115. (See also pp. 75, 98, &c.).

He supposes r to be 180 and r' to be successively 170, 160, 150, 140; f is $\frac{1}{2}$. Also, in the examples given, $n = 3$. If we make, for instance, $r' = 150$, we have

$$1 + x = \frac{180}{150};\ x = \frac{1}{5};\ w' = w\left(1 + \frac{1}{10}\right):$$

and the three fractions of the whole are $\frac{1}{6}$, $\frac{19}{36}$, $\frac{11}{36}$. The whole being 180, these are 30, 95, 55 as in Mr. Ricardo's example. Also the price is $\frac{6p}{5}$, where $p = £4$: whence the money value is easily calculated.

Since the sums paid as wages and the rate of wages differ at the two periods, the capital employed will differ. If, however, we suppose the value of the capital the same at the two periods, the rate of profit is altered in the proportion $\frac{n-1-fx}{n-1}$ If the capital be supposed to be agricultural produce and consequently its value to partake of the increase of prices, profits would be diminished in the proportion $\frac{n-1-fx}{(n-1)(1+x)}$. Compare Ricardo; p. 116.

PRICES.

11. In the above example, Art. 9, the price of corn is supposed to depend entirely on the labour directly employed to produce it. This is Mr. Ricardo's mode of treating the subject. It is, however, somewhat inaccurate according to his own principles: for in applying the postulate of price, we are to take into account the capital employed as well as the labour.

In consequence of this consideration, the prices of commodities will be effected by the proportion and durability of fixed capital requisite for their production. Mr. Ricardo has several propositions on this subject which may be included in the following formulæ.

Let l be the number of labourers who this year work by means of a machine or any other kind of fixed capital: l' the number of labourers who were employed last year in making

this machine: γ the rate of profit. The machine at the end of last year was worth the wages expended on it, together with profits, that is $(1+\gamma)l'w$. It must produce, this year, the profit on what it is worth, and the wages of those who work it, with profits: that is, it must produce $\gamma(1+\gamma)l'w+(1+\gamma)lw$.

In the same manner let a machine have been produced by the labour of l' labourers last year, l' in the preceding year, l'' in the year before that, and so on. And let l men be employed in working the machine; then its produce must be worth

$$(1+\gamma)lw+(1+\gamma)^2l'w+(1+\gamma)^3l''w+\&c.$$

Let L be the number of men requisite to obtain the same produce without the machine. Then Lw is their wages, and $(1+\gamma)Lw$ the value of the produce. Therefore, if the machine can be employed without loss

$$(1+\gamma)lw+(1+\gamma)^2l'w+(1+\gamma)^3l''w+\&c. = \text{or} < (1+\gamma)Lw,$$

$$\text{and } l+(1+\gamma)l'+(1+\gamma)^2l''+\&c. = \text{or} < L.$$

$$\text{Hence } l+l'+l''+\&c. < L;$$

or, when machinery is employed, it has always cost less labour than would obtain the same produce without machinery, Ric. p. 44.

Let b be the sum expended upon an average, every year for k years, in the construction of the machine, a the value of the machine when constructed. Then

$$c=b\{(1+\gamma)+(1+\gamma)^2\ldots+(1+\gamma)^k\}=b\frac{(1+\gamma)^{k+1}-1}{\gamma}.$$

12. When a machine, or stock of any kind, is employed in production, a certain sum is annually requisite to replace the wear and tear of the capital so existing. Mr. Ricardo supposes these annual sums to accumulate at interest, so as to become suf-

ficient to purchase a new machine just when it is wanted. Let k be the duration of the machine; b the annual reserve for it; therefore we must have an annuity b for k years, of which the present value is c. Therefore

$$b = \frac{c\gamma\,(1+\gamma)^k}{(1+\gamma)^k-1} = \frac{c\gamma}{1-(1+\gamma)^{-k}}.$$

$$\text{Let } b = \frac{c}{d};\ \therefore \frac{1}{d} = \frac{\gamma}{1-(1+\gamma)^{-k}};\ d = \frac{1}{\gamma} - \frac{1}{\gamma\,(1+\gamma)^k}.$$

As k increases $\frac{1}{\gamma\,(1+\gamma)^k}$ decreases, and d increases. For instance, if $\gamma = \frac{1}{10}$, when $k=1$, $d = \frac{10}{11}$, and when $k=10$, $d=6{,}14$.

Now we have

$$pr = lw\,(1+\gamma) + b + \gamma c,$$

$$= lw\,(1+\gamma) + \frac{c}{d} + \gamma c:$$

or if $c = mlw$,

$$pr = lw\,\{1+\gamma + \frac{m}{d} + \gamma m\}.$$

Let wages rise to $(1+u)w$: and let p become in consequence $(1+x)p$; then

$$(1+x)\,pr = lw\,\{(1+\gamma')\,(1+u) + \frac{m}{d} + \gamma' m\}$$

subtracting, $xpr = lw\,\{u(1+\gamma') - (\gamma-\gamma')\,(1+m)\}$;

And it is manifest that u being given, and γ' thence determined, the value of x will be least in the cases where m is the greatest; that is, the rise of price is greatest in the cases where the proportion of fixed to moveable capital is greatest.

The rise of price is here estimated in labour. But if we suppose money to be produced always by an equal quantity of

unassisted labour, we may measure by it the variations of price in other things.

Let λ labourers, unassisted by capital, produce ρ units of money. When wages rise, the produce of the same labour will be of the same value, therefore the unit of money will not alter in value. Hence we have

$$\rho=\lambda w(1+\gamma),\ \rho=\lambda w(1+u)(1+\gamma')$$

$$(1+\gamma')(1+u)=1+\gamma;\ 1+\gamma'=\frac{1+\gamma}{1+u};$$

$$\gamma-\gamma'=\frac{(1+\gamma)u}{1+u}\text{: hence by what precedes}$$

$$xpr=lw\left\{\frac{u(1+\gamma)}{1+u}-\frac{(1+\gamma)(1+m)u}{1+u}\right\}=-\frac{mulw(1+\gamma)}{1+u}.$$

This is always negative, and hence we have Mr. Ricardo's assertion that ***on these suppositions,*** prices fall by a rise of wages. (p. 41).

The above calculation is inexact, for d would alter when γ becomes γ': but if d be considerable, this error would be inconsiderable.

The supposition here made, that profits are entirely passive, is manifestly inadmissible; for if profits were 10 per cent., a rise of 10 per cent. in wages would annihilate profits altogether: which is absurd.

TAXES.

13. The general problem on this subject, is to determine the ultimate incidence of any given tax, according to the preceding postulates.

(1). ***Taxes on raw Produce*** (Ric. Chap. VIII).

Let a tax which is a portion t of the whole, be imposed on the produce. It affects all soils, and therefore the limiting soil,

on which the produce is r. Let the price before the tax be p and after it $p(1+x)$; the rest of the notation as before. Hence $(1-t)(1+x)pr$ is now the value of the cultivator's portion of the produce, which must pay wages and profits. Also profits remain unchanged, for if they were to fall, the cultivator would remove his capital to other employments. Hence

$$pr = lw + \gamma c, \quad pr(1-t)(1+x) = lw(1+fx) + \gamma c$$

subtract, and $pr(1-t)(1+x) - pr = lw.fx$: whence

$$x = \frac{prt}{pr(1-t) - flw} = \frac{nt}{n(1-t) - f}.$$

If, as before, $n=3$, $f=\frac{1}{2}$, and if $t=\frac{1}{10}$, $x=\frac{3}{22}$; prices rise as 11:14. In these cases prices rise so as to compensate the whole of the cultivator's loss and of the increased expenses of the labourer. The whole tax (assuming our Postulates) falls upon the rich consumer.

This is true on the supposition that the consumption is not at all affected by the price. Col. Thompson has properly observed that this assumption is by no means correct; and has shewn that if we make the more natural supposition that an increase of price will diminish the consumption, Mr. Ricardo's conclusions are no longer true. Pursuing Col. Thompson's views, I have, in a preceding memoir, given formulæ expressing the portions of such a tax which fall respectively in Rent, Profits, and Prices, and have shewn that in general the former is the largest portion. (*Camb. Trans.* Vol. III).

According to the postulate of wages, no part of such a tax could fall on wages.

The postulate of wages is however, as I have already said, perfectly gratuitous; and there does not appear to be any good reason for believing that a tax upon raw produce, whether food, or any other articles, might not fall to a great extent on the wages of the labourer.

The effect of a change of wages, such as is here supposed, upon foreign trade, will appear from what we shall have to say in treating of that part of the subject.

2. *Taxes on Rent*, (Ricardo, Chap. VIII.)

If r be the produce of the limiting soil, and r_1 of any other soil, we shall have, after the tax as before, $pr=lw+\gamma$: and as γ, r, c, l, w, remain the same, p is not altered. Hence if a tax is laid on rent to the amount of the fraction t, it will fall wholly on rent: which will be reduced from pr_1-pr to $(1-t)(pr_1-pr)$.

If what is called rent, be in part interest for capital, a tax on rent will be governed by different laws, so far as that portion is concerned.

3. *Land Tax*, (Ricardo, Chap. X.)

Let T be the amount of the tax per acre. Therefore $pr'-T$ is the cultivator's portion of the produce of the limiting soil; γ will not be changed; therefore

$$p'r-T=lw'+\gamma c, \quad pr=lw+\gamma c,$$

$$(p'-p)r-T=l(w'-w); \quad prx-T=lwfx; \quad x=\frac{T}{pr-flw}=\frac{T}{(n-f)lw};$$

the tax will wholly affect the price, and the results will be exactly the same as in a tax on raw produce.

4. *Taxes on Profits*, (Ricardo, Chap. XIII.)

These taxes are supposed to be partial, that is, to be levied only on certain employments of capital. In that case they will

not diminish the rate of profits, because if they did, the capital would be transferred to other employments.

As before, let r be the number of units produced by a total capital c, (fixed and circulating), p the price of each unit. Then $pr=lw+\gamma c$: profits are γc. If the tax be a fraction t, γ will remain the same after its imposition. Let the price become p'; therefore

$$p'r=lw+\gamma c+t\gamma c.$$

$$\text{Hence } \frac{p'}{p}=1+\frac{t\gamma}{lw+\gamma c},$$

the productions of those employments of capital in which profits were taxed, would rise in price, compared with those in which the tax was not applied. If money be not taxed, the money prices of such commodities will rise.

It has been seen, (Art. 12.) that if a person employ a fixed capital $c-lw$, which is replaced in k years, and a circulating capital lw in paying wages, we must have

$$pr=lw+b+\gamma c, \text{ where } b=\frac{\gamma(c-lw)}{1-(1+\gamma)^{-k}},$$

in this case profits are γc.

If we now suppose profits to be taxed, and money not to be taxed, money prices will rise. Mr. Ricardo supposes that in this case the money value of the profits will remain the same as before, (p. 272. 1st ed.).

Therefore $p'r=lw+b+\gamma c+t\gamma c$;

$$\therefore \frac{p'}{p}=1+\frac{t\gamma c}{lw+b+\gamma c}.$$

Let the fixed capital be m times the circulating, or $c-lw=mlw$; hence $c=(m+1)lw$. Also let $b=\frac{mlw}{d}$, where it is clear that d is greater as k is greater (see Art. 12.), then

$$\frac{p'}{p} = 1 + \frac{t\gamma(m+1)}{1 + \frac{m}{d} + (m+1)\gamma}$$

$$= 1 + \frac{t}{\dfrac{1 + \frac{m}{d}}{\gamma(m+1)}}.$$

Hence it appears that the increase of price on these suppositions will be greater according as $\dfrac{1+\frac{m}{d}}{1+m}$ is less.

$$\text{But } \frac{1+\frac{m}{d}}{1+m} = 1 - \frac{1-\frac{1}{d}}{1+\frac{1}{m}}.$$

Therefore (d being >1) the rise of price from the tax will be greater in commodities, as m is greater, and as d is less; that is, as the *proportion* of fixed capital is greater, and as its *durability* is less, (Ricardo, p. 273).

If we suppose two cases, in one of which $m=4$, in the other $m=\frac{1}{4}$ (Ricardo, p. 273.), and if we neglect $\frac{m}{d}$, we have, γ being $=\frac{1}{5}$,

$$\text{in the first case } \frac{p'}{p} = 1 + \frac{t}{2},$$

$$\text{in the second } \frac{p'}{p} = 1 + \frac{t}{5}.$$

The rise of price is $2\frac{1}{2}$ times as great in one case as in the other.

A tax on all profits is an income tax, and the effect of such a tax will thus be unequal on the prices of different commodities.

The preceding calculation is upon the supposition that the money metal is produced in the country in which the tax is laid. If the precious metals be imported from other countries, the investigation of the consequences will require considerations which will hereafter be stated in treating of foreign trade.

5. *Taxes on Wages.* (Ric. Chap. XIV.)

We assume still the postulate of wages, although, as has been said, it appears to be unfounded, and consequently the inferences to which it leads, as to the incidence of taxes, will not be verified in fact. But, granting the principle, there appears to be some difficulty in tracing its results, and Mr. Ricardo, in attempting to do so, has been led into an arithmetical fallacy, as was pointed out by Col. Thompson, and as I have shewn in a preceding Memoir. (Ricardo, p. 301, and *Camb. Trans.* Vol. III. p. 192). Mr. Ricardo's opinion was, that taxes upon wages fall entirely on profits, while other writers maintain that they will affect prices and rent. Calculations such as we have already employed will enable us to obtain the true result in this case.

Let the labourer expend a fraction f of his wages on agricultural produce (as corn), and the remainder $1-f$ on manufactured goods. And by the postulate of wages, let his real consumption remain the same after the imposition of the tax, as it was before. If prices rise, the rise may be different in corn and in goods, according to the proportion of fixed and circulating capital in the two cases.

Suppose that for corn the fixed capital $c-lw=mlw$: for goods in the same manner let $\mu\lambda w$ be the fixed capital: and let the wear and tear of the fixed capital be included in the circulating capital. Also suppose that after the imposition of the tax, the price of

corn rises in the proportion $1 : 1+x$, the price of goods in the proportion $1 : 1+\xi$; and that wages rise in the proportion $1 : 1+u$. Therefore the wages of the labourers employed in producing corn become $lw(1+u)$; and we have by the condition of remunerating price,

$$pr = lw(1+\gamma) + \gamma m l w;$$

$$(1+x)pr = lw(1+u)(1+\gamma') + \gamma' m l w; \text{ whence}$$

$$1+x = \frac{(1+u)(1+\gamma') + \gamma' m}{1+\gamma+\gamma m}; \text{ similarly}$$

$$1+\xi = \frac{(1+u)(1+\gamma') + \gamma'\mu}{1+\gamma+\gamma\mu}.$$

Now by supposition the labourer's consumption of corn and goods is the same as before; and therefore the expense of corn, which was fw, becomes $f(1+x)w$. Similarly the expense in goods, which was $(1-f)w$, becomes $(1-f)(1+\xi)w$: and these portions together with the tax make up the whole of the wages $(1+u)w$. Let the tax be a portion t of the whole. Then

$$f(1+x) + (1-f)(1+\xi) + t(1+u) = 1+u, \text{ or}$$

$$\frac{f(1+u)(1+\gamma') + f\gamma' m}{1+\gamma+\gamma m} + \frac{(1-f)(1+u)(1+\gamma') + (1-f)\gamma'\mu}{1+\gamma+\gamma\mu} = (1-t)(1+u).$$

From this equation γ' being known, u is known, or *vice versa*.

It appears from the result that the problem of the incidence of a tax on wages, under the conditions here supposed, is indeterminate. A part falls on the consumer in raised prices, a part on the capitalist in diminished profits; and the principles hitherto assumed do not enable us to determine the respective amount of these portions. In fact we suppose both the consumer and the capitalist to be entirely passive, so as to have no power of throwing their loss on any other person, and on this supposition

it is manifest, that there is nothing to disturb the equilibrium, whatever be the amount of this loss in either case.

It will be seen however, that the result which Mr. Ricardo appears to have inferred, viz. that the whole tax *in each case* would be taken from profits, cannot possibly be true. For in such a case it would differently affect profits in different employments, according to the proportion of fixed capital: and hence some succeeding adjustment must necessarily take place, in order to bring back profits to an equality.

1. Let it be supposed that profits are unaltered by the tax. Therefore

$$\gamma' = \gamma, \quad 1+x = 1 + \frac{(1+\gamma)\,u}{1+\gamma+\gamma m},$$

$$x = \frac{(1+\gamma)\,u}{1+\gamma+\gamma m}, \quad \xi = \frac{(1+\gamma)\,u}{1+\gamma+\gamma\mu}.$$

$$\text{Also } f(1+x)+(1-f)\,(1+\xi)+t(1+u)=1+u,$$

$$\text{gives } t=(1-t)\,u-fx-(1-f)\,\xi$$

$$= \left\{1-t-\frac{f(1+\gamma)}{1+\gamma(m+1)}-\frac{(1-f)\,(1+\gamma)}{1+\gamma\,(\mu+1)}\right\} u.$$

whence u is known.

If, as an extreme case, we suppose no fixed capital in agriculture, and no circulating capital in manufactures, $m=0$, $\mu=\infty$, whence

$$x=u, \ \xi=0. \quad fu+t+tu=u, \ u=\frac{t}{1-t-f}.$$

If $t=\frac{1}{10}$, $f=\frac{1}{2}$, we have $u=\frac{1}{4}$. A tax of $\frac{1}{10}$ on wages would raise the price of agricultural produce $\frac{1}{4}$, while manufactures remained unaltered; and would raise wages in the same proportion.

2. Let it be supposed that prices in any particular employment, as manufactures, are not effected by the tax; therefore $\xi=0$. Therefore

$$(1+u)(1+\gamma')+\gamma'\mu=1+\gamma+\gamma\mu; \text{ hence}$$

$$\gamma'=\frac{\gamma(\mu+1)-u}{\mu+1+u};\quad 1+\gamma'=\frac{(1+\gamma)(\mu+1)}{\mu+1+u}.$$

$$\text{Hence, substituting, } x=\frac{(1+\gamma)u(\mu-m)}{(\mu+1+u)(1+\gamma+\gamma m)}.$$

And the equation $f(1+x)+(1-f)(1+\xi)+t(1+u)=1+u$, gives, in this case, $fx+t+tu=u$, whence

$$t=u-tu-fx$$

$$\text{or } t=\left\{1-t-\frac{f(1+\gamma)(\mu-m)}{(\mu+1+u)(1+\gamma+\gamma m)}\right\}u.$$

whence u may be determined by a quadratic, and hence γ'.

Let $m=\frac{1}{2}$, $\mu=2$, $\gamma=\frac{1}{10}$, $f=\frac{1}{2}$,

$$t=\left\{1-t-\frac{33}{46(3+u)}\right\}u.$$

3. Let the proportion of fixed to circulating capital be, on the *average*, m, so that $\mu=m$, $\xi=x$.

$$\text{Then } f(1+x)+(1-f)(1+x)+t(1+u)=1+u$$

$$\text{or } t+tu+x=u$$

$$t+tu+\frac{u(1+\gamma')-(m+1)(\gamma-\gamma')}{1+\gamma+\gamma m}=u.$$

If we suppose, moreover, that prices remain the same after the tax,

$$x=0,\quad t+tu=u,\quad u=\frac{t}{1-t}.$$

Also $u(1+\gamma')-(m+1)(\gamma-\gamma')=0$; $\gamma'=\frac{(m+1)\gamma-u}{m+1+u}$.

If $t=\frac{1}{10}$, $u=\frac{1}{9}$; a tax of $\frac{1}{10}$ raises wages $\frac{1}{9}$.

Also for the effect on the rate of profit, we have

$$\gamma-\gamma'=\frac{u(1+\gamma)}{m+1+u}$$

profits are less affected, as (m) the proportion of fixed capital, is greater.

FOREIGN TRADE.

14. We now come to a part of the subject altogether distinct from that which we have been considering; a part both of great interest and great difficulty: I refer to the doctrines concerning foreign trade; and, as connected with these, the laws of the influx and efflux of the precious metals, and the influence upon prices exercised by their abundance or scarcity. This is a portion of Political Economy on which the postulates which have hitherto been the basis of our reasoning have no bearing. The proportionality of the exchangeable value to the cost or labour of production no longer obtains, when the labour of different countries is concerned. "The produce of the labour of 100 Englishmen may" as Mr. Ricardo says, "be given for the labour of 80 Portuguese, 60 Russians, or 120 East Indians." Nor can we assume the equality of profits in different countries. The difficulty with which labour and capital travel from one country to another, is a sufficient obstacle in the way of the establishment of such a uniformity of the value of labour, and of the rate of profits.

I shall have to state some of the principles which are said to apply to the questions now before us; they have been put forwards with great ingenuity by Mr. Ricardo and his followers; and several curious speculations have been founded upon them, especially those connected with the influence of the manufacturing skill of a country upon the influx of gold into it, and consequently upon the general scale of prices.

It will be seen, that besides the principles which I borrow from Mr. Ricardo, I shall have to make several assumptions for the sake of reducing the problems before us to calculation. It will of course be understood that these assumptions are not intended to be maintained as exactly, or even as approximately true. Their use is to shew how the numerical examples which serve as illustrations of the principles, may be properly and consistently generalized and limited. But though these investigations cannot pretend at present to much precision, it is hoped that they may serve to shew, of what kind and how many are the data on which the exact solution of such problems must depend: and they may thus be of some use in directing future attention with regard both to the laws and the quantities involved in these difficult questions.

I suppose the Commerce of different countries to take place free from all prohibitions, duties, drawbacks, &c. This simplification appears to be the easiest mode of undertaking the investigation at first. I may, perhaps, on some other occasion consider similar problems with the additional conditions which such restrictions introduce.

On this supposition we shall assume the following principles.

POSTULATES CONCERNING TRADE.

I. The merchant will buy a commodity at one place, and sell it at another, if by so doing he can recover the money cost with the profit usual in his own country. The consumer will buy a commodity of foreign rather than the same commodity of domestic production, if by so doing he can obtain it cheaper.

This is the fundamental principle of all foreign trade, all restrictions being, as has been said, supposed to be removed.

II. When a country has as much *gold* (or any other money-metal) as is wanted to circulate her commodities; if an additional quantity of gold be imported and retained in the country, the circulation of commodities remaining the same, money prices will rise.

Gold being more abundant, will of course become less valuable; that is, commodities will be more valuable, if compared with gold, or gold-prices will rise.

I shall suppose, in what follows, that prices rise *in proportion* to the increased quantity of gold: that is, if g be the original quantity, and h the quantity imported, prices will rise in the ratio $g : g+h$. If p be the price at first, $p\left(1+\frac{h}{g}\right)$ is the price at last.

If however gold is imported in consequence of a balance of exports over imports, it is probable that the internal trade would increase, and the circulation of commodities in the country would be greater than when there were no exports or imports. Hence a greater quantity of gold than g, the original quantity, would now be wanted, and the price would not rise so much as has just been mentioned. Instead of $p\left(1+\frac{h}{g}\right)$ it might become

$p\left(1+\frac{h}{g'}\right)$ when g' is the quantity of gold requisite for the internal circulation of the country in its new condition. If we suppose $g'=g+mh$ where m is a fraction; the price at last

$$cp'=p\left(1+\frac{h}{g+mh}\right)=p\left(1+\frac{\frac{h}{g}}{1+m\frac{h}{g}}\right).$$

Or more generally we may suppose $p'=p\left(1+\phi\frac{h}{g}\right)$, when $\phi\frac{h}{g}$ is some quantity depending on $\frac{h}{g}$.

It appears to be very difficult to find any criterion or measure of the quantity of gold requisite for the independent interior circulation of a country: consequently it is extremely difficult to ascertain the effect on prices produced by the country having more than this quantity, which, as will be seen, is the result of a superiority in manufacturing skill.

It is manifest, however, that the quantity of gold requisite for circulation in any country is diminished by any contrivance which introduces any other circulating medium, as paper; or which increases the velocity and facility of any portion of the circulation. The value of the paper (if convertible) depends immediately on the value of the gold; but the *quantity* of gold *wanted*, may depend on the use and quantity of paper; and the quantity of gold wanted in a country is one of the conditions which affects its value there relatively to that in other countries.

If a part of the currency of the country be paper, the effect on prices produced by the introduction of superfluous gold will be more difficult to estimate. For the sake of simplicity how-

ever I shall still suppose the effect proportional to the quantity introduced. If g_1 be the money at one period, and g_2 the same with the addition of the imported gold, I shall suppose that prices increase in the ratio $g_1 : g_2$.

The preceding considerations may seem to shew how difficult and complex must be all calculations on this subject.

III. In the condition of equilibrium of a trading country, the annual exports and imports must be equal in money value.

We speak here of a condition of things which may continue from year to year unaltered. It is clear that in such a condition we could not have a regular excess of exported value over imported, or the reverse. For if we had the former case, the balance must be brought here in gold, and we should import a fixed quantity of gold every year. But in consequence of this circumstance, the prices of all our commodities would rise, some that had been usually exported would cease to be so from their increased price, and this operation would never stop till the exports had reached a point at which their money value was equal to that of the imports.

Let p be the price of a unit of any exported article, e the number of units exported: similarly let p', p'', &c. be other prices, e', e'', &c. the corresponding quantities: then $ep+e'p'+e''p''+$&c. is the value of the whole export. This may be represented thus Sep. In the same manner if i, i', i'', &c. be the quantities imported of articles of which the prices are s, s', s'' &c. $is+i's'+i''s''+$&c. $=Sis$ represents the whole of the imports. Hence we have by the principle just explained $Sep = Sis$.

IV. The prices supposed here are the remunerating prices in the equilibrium of demand and supply.

Let p be the price at which cloth can be imported into Chili from England, with English profits. Then it is here supposed that p is the price at which it is there sold, though it may for a time be sold for a higher price from the demand being active, or for a lower price, the supply being excessive. The gradual extension of demand, and the fluctuations and miscalculations of supply, though causes of perpetual and powerful operation, are left out of consideration, as not effecting that condition of equilibrium to which our investigations here refer. The estimation of these alternations, indeed, appears to be a fitter employment for the sagacity of the practical merchant, than for the reasoning of the theoretical economist. To apply mathematical formulæ to them would probably not be a more successful undertaking than it would be to calculate on mechanical principles, the alternation of smaller and larger surges in the progress of a rising tide.

15. I shall now proceed to determine mathematically the conditions and consequences of an export and import trade.

Let p be the price of a unit of a commodity manufactured *here* and carried into a foreign country, p' the price at which it could be manufactured in the foreign country: then if $p<p'$ the article will be exported. Let e be the quantity exported, then ep is the value exported of such a commodity: and Sep, the sum of all such products, is the whole exported value.

Similarly, if s be the value of a commodity imported, s' the price at which it could be manufactured here, it will be imported if $s<s'$. And i being the quantity, Sis is the whole imported value.

If Sep is not $=Sis$, the balance $Sep-Sis$ must be imported in gold, which will thus be added to the quantity of gold

existing in the country. Hence prices will be altered, and with them the quantity of exports.

Let e_1, $e_2 \ldots e_t$ be the exports in the 1st, 2d... t^{th} year,

p_1, $p_2 \ldots p_t$, the corresponding prices,

g_1, $g_2 \ldots g_t$, the quantity of gold (or other currency) in the country at the beginning of these years, (the amount of the currency being supposed to change only in consequence of the importation of gold).

Also let it be supposed that the increase of price is in proportion to the increase of the quantity of gold, (see Post. II.;)

$$\text{that is, } p_t = p_1 \frac{g_t}{g_1}.$$

We have therefore, since $g_{t+1} - g_t$ is the gold imported in the tth year, which must equal the excess of exports, $Se_t p_t - Sis = g_{t+1} - g_t$, or, putting for p_t its value,

$$\frac{p_1 g_t}{g_1} Se_t - Sis = g_{t+1} - g_t;$$

$$\text{therefore } g_{t+1} = (1 + \frac{p_1}{g_1} Se_t) g_t - Sis \ldots (1).$$

If we know the law of the diminution of the annual exports Se_t, we have Se_t in terms of t, and hence, by integrating this equation of finite differences, we can easily find g_t in terms of t, and hence the quantity of gold in the country after any number of years. In practice (on the suppositions already made) g_t would tend rapidly to a limit, and the quantity of gold after a few years would receive no sensible increase. If t be the number of years which gives the limit with sufficient accuracy, we have

$$Se_t p_t - Sis = 0;$$

$$\therefore \frac{p_1}{g_1} g_t Se_t - Sis = 0, \quad p_1 Se_t = \frac{g_1}{g_t} Sis \ldots (2).$$

16. It will be more simple to suppose the successive quantities of exports Se_1, Se_2, Se_3, &c. to be known. In this case the calculation will stand as follows.

Let $m_t = 1 + \frac{p_1}{g_1} Se_t = \frac{g_1 + p_1 Se_t}{g_1}$;

namely, the ratio of the gold originally in the country + the value of the exports in the tth year, to the original gold; the exports being reckoned at the original price. Also, let the exports $= ng_1$. Then we have, by equation (1)

$$g_{t+1} = m_t g_t - ng_1,$$

which may be easily reduced to calculation when the number of terms m_1, m_2, m_3, &c. is not large.

Let the exports and imports reach their equilibrium in *three* years; we then have the equations

$$g_2 = m_1 g_1 - ng_1$$

$$g_3 = m_2 g_2 - ng_1$$

$$g_4 = m_3 g_3 - ng_1.$$

Hence $g_4 = \{m_3 m_2 m_1 - (m_3 m_2 + m_3 + 1)n\} g_1 \ldots (3)$.

For instance, let the value of the imports be originally $\frac{1}{6}$ of the value of the currency of the country: and let the exports be, in the first year $\frac{1}{4}$ of the same value; in the two successive years let the exports be so diminished in quantity as to be $\frac{1}{5}$ and $\frac{1}{6}$ if estimated at the same price: therefore $Sp_1 e_1 = \frac{1}{4} g_1$, $Sp_1 e_2 = \frac{1}{5} g_1$, $Sp_1 e_3 = \frac{1}{6} g_1$.

Here $n = \frac{1}{6}$, $m_1 = \frac{5}{4}$, $m_2 = \frac{6}{5}$, $m_3 = \frac{7}{6}$.

Therefore by (3)

$$\frac{g_4}{g_1} = \frac{7}{4} - \left(\frac{7}{5} + \frac{7}{6} + 1\right)\frac{1}{6} = \frac{315}{180} - \frac{107}{180} = \frac{208}{180} = \frac{52}{45}.$$

Hence the quantity of money in the country is increased in the ratio 45 : 52, and therefore, by Post. II, the prices of all domestic commodities are increased in the same ratio. The prices of imported commodities are not affected by this change, for the foreigners will be content with a remunerating money price, and will be prevented by their own competition from obtaining more.

By last article, equation (2),

$$p_1 Se_4 = \frac{g_1}{g_4} Sis = \frac{45}{52} \cdot \frac{1}{6} g_1 = \frac{45}{312} g_1 = \frac{1}{7} g_1 \text{ nearly.}$$

The value of the ultimate exports and imports, at the original prices, would be nearly $\frac{1}{7}$ and $\frac{1}{6}$ of the original currency, and in the equilibrium of exports and imports these values become equal, by means of the increased quantity of gold in the country. To find the gold in the country at the end of each of the three years, we have

$$g_2 = \frac{13}{12} g_1; \quad g_3 = \frac{17}{15} g_1; \quad g_4 = \frac{52}{45} g_1;$$

Also for the value of the exports in each year,

$$Se_1 p_1 = \frac{1}{4} g_1; \quad Se_2 p_2 = \frac{g_2}{g_1} Sp_1 e_2 = \frac{13}{60} g_1; \quad Se_3 p_3 = \frac{g_3}{g_1} Sp_1 e_3 = \frac{17}{90} g_1.$$

If we suppose the skill of the workmen of this country to increase, so that the commodities which we can manufacture cheaper than foreigners become more numerous, the value of the exports (restrictions being supposed absent) will increase. In that case such an importation of gold, and consequent increase of all prices here, will occur, as has been supposed in the last and present article.

THE RATE OF EXCHANGE.

17. The different tendency of the money metals to flow into or out of different countries, and the different level of prices consequent on this, is not at all indicated by the rate of exchange. On the contrary when the equilibrium is attained which such tendencies produce, the exchange is at par. The average rate of exchange can only be for or against a country when the value of the exports is greater or less than that of the imports. In that case the following appear to be the principles upon which, in accordance with the views commonly promulgated on this subject, the amount of the rate must be determined.

POSTULATES CONCERNING EXCHANGE.

I. When between two countries *A* and *B*, the debts due from *A* to *B* are of greater money amount that those due from *B* to *A*, bills in *A* payable in *B* will be sold for more than their nominal value.

The seller in *A* of a bill on *B*, authorizes a person coming from *A* to make a demand on his correspondent in *B*. If his correspondent cannot answer this demand by a bill on *A*, he must pay in coin; and if there be not, arising from mercantile transactions, a sufficient number of bills on *A* to meet such demands, gold must be exported from *A* to *B* for that purpose. The seller of the bill in *A* will not therefore sell it without providing for the foreseen expense of this transfer of gold, and the price of the bill will rise above its nominal value, by some *premium* dependent on the amount of such expense supposed to be necessary.

II. The aggregate amount of all the premiums of bills on *B* sold in *A* (supposing them to have the average value) will be the expense of the transfer of so much gold as is necessary to restore the equilibrium.

If the amount of the premiums were greater, it would be possible to make money by drawing bills on *B*, selling them, and transferring gold from *A* to meet them. If the amount of the premiums were less, the holders of bills would refuse to sell, and those who had debts to discharge in *B* would have to incur the expense of transferring gold: and, rather than do this, they would pay a larger premium.

III. The premium on all good bills on *B* sold in *A* will be the same.

Supposing the market of bills regulated by competition this will obviously be true: for one good bill is on the same footing as another.

18. I now proceed to calculate the rate of exchange under given conditions of the imports and exports.

I shall suppose that the bills on *B* negociated in *A* are drawn for the payment of the exports from *A* to *B*. Let *Sep* be the exports, and *Sis* the imports, in this case. Then *Sep* is the amount of the bills drawn, and $Sep - Sis$ of the gold which it will be necessary to transfer, in order to balance the account. Let x be the premium on 1. Therefore $xSep$ is the premium on the whole mass of bills. Also, let y be the expense of transferring a unit of gold from *B* to *A*. Therefore $y(Sep-Sis)$ is the expense of transferring the balance. Hence, by postulates II and III of last article, we have

$$xSep = y(Sep - Sis);$$

$$\therefore\ x = \frac{Sep - Sis}{Sep} y$$

whence x the rate of exchange is known.

19. In the case assumed in Art. 6, we may now calculate the rate of exchange during the progress of the alteration.

Let $x_1\ x_2\ x_3$ be the rates of exchange during the 3 first years,

$$x_1 = \left\{1 - \frac{Sis}{Se_1 p_1}\right\} y = \left\{1 - \frac{2}{3}\right\} y = \frac{1}{3} y$$

$$x_2 = \left\{1 - \frac{Sis}{Se_2 p_2}\right\} y = \left\{1 - \frac{30}{39}\right\} y = \frac{3}{13} y$$

$$x_3 = \left\{1 - \frac{Sis}{Se_3 p_3}\right\} y = \left\{1 - \frac{15}{17}\right\} y = \frac{2}{17} y$$

$$x_4 = \left\{1 - \frac{Sis}{Se_4 p_4}\right\} = 0.$$

If the expense of transferring gold be 1 per cent., the exchange will be, in the 1st, 2nd, 3rd year respectively, $\frac{1}{300}$, $\frac{1}{433}$, $\frac{1}{850}$, and afterwards 0.

20. I will not quit the subject without again stating that I by no means wish to attribute any mathematical certainty, or any extraordinary value, to the preceding results, considered with reference to their application to the real circumstances of human affairs. I must however observe, that if they are useless and inapplicable, the fault resides in the postulates which I have borrowed from Mr. Ricardo and others, and not in the mode of deducing the consequences of these principles. If these postulates

were universally and strictly true, the results, as above stated, would be exactly, and in all cases, verified: I may add, that if we had reached such a point in the progress of this science—if it were reduced to a few certain principles, and to a long train of deductions from these—(a form which it appears to have been the object of Mr. Ricardo and his followers to give to it—) the mathematical method would be the one proper for its treatment, being the most certain, the shortest, and, with a little preparation, the simplest. Mathematics is the logic of quantity, and will necessarily, sooner or later, become the instrument of all sciences where quantity is the subject treated, and deductive reasoning the process employed.

I am however well aware, that the pretensions of Political Economy to such a scientific character, are as yet entirely incapable of being supported. Any attempt to make this subject at present a branch of Mathematics, could only lead to a neglect or perversion of facts, and to a course of trifling speculations, barren distinctions, and useless logomachies. "Collocatio ejus inter mathematica" as Bacon says of another science, "hunc ipsum defectum et alios similes peperit; quia a phenomenis præmature discessum est." And these defects may be incurred, even though common verbal reasoning be substituted for mathematics, if the course adopted be that of assuming principles and definitions, and making these the origin of a system. The most profitable and philosophical speculations of Political Economy are however of a different kind: they are those which are employed not in reasoning *from* principles, but *to* them: in extracting from a wide and patient survey of facts the laws according to which circumstances and conditions determine the progress of wealth, and the fortunes of men. Such laws will necessarily at first, and

probably always, be too limited and too dependent on moral and social elements, to become the basis of mathematical calculation: and I am perfectly ready to admit, that the discovery of such laws, and the investigation of their consequences, is an employment of far higher philosophical dignity and importance than any office to which the Mathematician can aspire.

W. WHEWELL.

TRINITY COLLEGE,
May 7, 1831.

ERRATA.

Page	Line	
158	9	*for* extend *read* obscure.
173	7	from bottom, *for* effected *read* affected.
175	4	————, *for* greatest *read* least.
180	7	*for* less *read* greater.
—	9	*for* less *read* greater.
192	8	*for* exports *read* imports.

Addition to a Paper "On the Nature of the Light in the Two Rays produced by the Double Refraction of Quartz."

By G. B. AIRY, M.A.; M.G.S.;

LATE FELLOW OF TRINITY COLLEGE; PLUMIAN PROFESSOR OF ASTRONOMY AND EXPERIMENTAL PHILOSOPHY IN THE UNIVERSITY OF CAMBRIDGE; AND FELLOW OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *April* 18, 1831.]

After I had made the experiments described in a paper which was read to this Society on *February* 11, 1831, I received from Mr. Dollond an apparatus constructed under my direction, which for convenience and extent of application exceeds any other that I have seen. The parallel rays that fall on a piece of plate glass blackened at the back are reflected, all completely polarized, and are received on the first lens, which makes them all pass through one point: then diverging they are received on the second lens, whose distance from that point is equal to its focal length, and they emerge from it parallel. In this state they are received on the analyzing plate (a piece of plate glass blackened behind) and all are of course completely analyzed. They are then received on the third lens (fixed in a sliding tube, like the eyeglass of a telescope) which makes them all pass through the eyehole. The lenses are of equal focal length, and their arrangement is precisely the same as that of the lenses in the old three-glass eye-

piece, measuring the distance between the second and third lenses along the incident and reflected ray. The analyzing plate, with the second and third lenses, turns on a spindle parallel to the rays polarized by the first plate, whose direction passes through the centers of the first and second lenses. To see the rings &c. produced by a crystal, it should be placed at the point where the rays cross between the first and second lens: and a specimen $\frac{1}{20}$ inch broad so placed will shew the rings with perfect brilliancy and clearness in their utmost extent. If it is wished to use a micrometer, the micrometer should be placed between the polarizing plate and the first lens, at a distance from the latter equal to its focal length, where it will be seen distinctly at the same time that the rings are seen distinctly. If it is wished to see the macled structure of quartz, amethyst, topaz, &c. the crystal is to be placed in the situation assigned for the micrometer: then no rings will be seen, but on turning the analyzing apparatus round its spindle the different parts will be differently coloured. For the use of plane-polarized light only, there is no need for a polarizing plate larger than the projection of the first lens: and the distance between the nearest edge of the polarizing plate and the first lens needs not to exceed by a large quantity the focal length of the latter. But as I now consider every apparatus incomplete which does not allow of the use of circularly and elliptically polarized light, I have had the polarizing plate separated so far from the lens that it allows Fresnel's rhomb (mounted as in Plate 8, fig. 4) to be placed between them, room being left for the micrometer &c. between the rhomb and the lens: and the polarizing plate is made so large that it will transmit plane-polarized light to the end of the rhomb, at whatever angle it is placed, while the other end is centrally opposite to the first lens.

To use artificial light with facility, a lamp is placed in the focus of a lens whose diameter is equal to the diameter of the circle described by the point of the rhomb. The board which carries the lamp and lens is cut at such an angle that on applying its sloped end to the side of the board carrying the other apparatus, the light is reflected by the polarizing plate in the proper direction without any farther adjustment.

The advantages of this apparatus are, that with a very small specimen, by day or by night, it will exhibit all the phænomena of plane and elliptically polarized light to the greatest angular extent that the eye can receive, and with conveniences of measurement that have never before been given: that thus a macled crystal, or a piece of unannealed glass, &c. can be as it were dissected by successive examination of different small portions: that any accidental roughness or inequality of the crystal produces no sensible effect: and that by placing the specimen in another position, the macled structure can be exhibited with singular clearness.

In the examination of the phænomena which quartz presents when exposed to circularly polarized light, detailed in my paper before alluded to, I found very great difficulty in consequence of the contraction of the field of view when Fresnel's rhomb is used with the common polarizing apparatus. In examining the two spirals inwrapping each other, I could see little more than a single line at a time: and it was only by carefully turning the crystal that I could discover the relation of one line to another. It was in fact principally to overcome this difficulty that I devised the apparatus here described: with this I can at once see the folds of the spirals as far as the colours are sensible. I have much satisfaction in seeing that my delineation was quite correct.

On considering my hypotheses relative to the nature of the two rays of quartz, the following method suggested itself as a means of verifying one part of the hypotheses, and as affording a power of measuring the ellipticity of the rays. Suppose (by placing Fresnel's rhomb in a position between 0° and 45°, or between 90° and 135°) elliptically polarized light is made to pass through quartz. Whether this be right-handed or left-handed, there is one direction (*A*) in which one ray of the quartz (suppose for instance the ordinary) is of just the same kind as the incident elliptical light. Consequently that light furnishes no extraordinary ray. Now if we take a direction (*a*) nearer to the axis by the smallest possible angle, and another (*b*) further from the axis by the smallest possible angle, than the direction just mentioned (*A*), the same elliptical light incident in the directions (*a*) and (*b*) will furnish extraordinary rays, but the paths of these extraordinary rays will differ (independently of all other causes) by half the length of a wave. For the elliptical light which is of the same kind as the ordinary ray in (*A*) is more elliptical than the ordinary ray in (*a*) and less so than that in (*b*). And therefore when we separate the elliptical light into an ordinary and an extraordinary ray in (*a*), it is the *defect* of its minor axis which produces the extraordinary ray: when we do the same for (*b*), it is the *excess* of its minor axis which produces the extraordinary ray. The vibration therefore which produces the extraordinary ray in (*a*) being in the positive direction, that which produces the extraordinary ray in (*b*) will be in the negative direction; or *vice versa*. And this amounts to the same as retardation or acceleration by half the length of a wave. It will readily be seen that this is independent of the crystalline separation of the two rays, and is true however small be the angle

between the directions (*a*) and (*b*), provided that one be nearer to the axis, and the other further from it; than (*A*). Now it is well known that the order of the rings depends on the number of lengths of wave which the extraordinary ray is in advance or retardation of the ordinary ray. At this place the advance or retardation is suddenly altered by half a wave. Consequently the order of the rings is suddenly altered by half an order: the rings become faint, and then there is a *saltus* of half an order in the colours. And the direction of the ray where this *saltus* takes place being observed, it gives the direction in which the ordinary ray has the same ellipticity as the incident light, which is known from the position of the rhomb.

By this reasoning I had satisfied myself that the relation between the direction of the ray and its ellipticity could be made evident to the eye. On trying it with the apparatus just described, I found that the appearance was exactly what I expected. On placing the rhomb in position 315°, the spirals are perfect. On turning it forwards, the internal folds break successively, (if the crystal be left-handed) in a line nearly horizontal, and the upper part of each unites itself with the lower part of the fold next beyond it. This continues, the outer folds being broken after the inner ones, till when the rhomb has reached 0° the appearance is that of perfect circles. On turning still forwards, the successive circles break (beginning with the outer ones) in a line nearly vertical: and when the rhomb has reached the position 45°, the spirals are perfect as before, but twisted 90°. Thus at any position of the rhomb intermediate to 0°, 45°, &c. one or more of the inner circles are complete, but distorted, and the exterior circles are changed to two inwrapping spirals. It is plain that the points where the spirals begin are the points where the

elliptical light supplies only the ordinary ray, as the colours one quarter of an order within these points, and one quarter of an order beyond them, are the same.

The conclusion, that the inner rings will be circles, and the outer ones two spirals, follows easily from this consideration. The intromitted elliptical light has a smaller minor axis than the ordinary ray (which is the only one that it resembles) in the directions nearest to the axis of the crystal, and therefore it will produce curves analogous to those produced by plane-polarized light; that is, analogous to circles. But it has a larger minor axis than the ordinary ray, in the directions far from the axis of the crystal, and therefore it will produce curves analogous to those produced by circularly-polarized light; that is, analogous to two spirals infolding each other. There is no difficulty in making a more accurate investigation, on the principles described in my former Paper: but I have not done it for a reason that will shortly appear.

I have not yet had the opportunity of making measures which are sufficient to point out the law that connects the ellipticity of the rays with the angle that they make with the axis. The following points however are made out. One of the rays certainly is right-handed elliptical, and the other certainly left-handed elliptical (or so nearly, that no difference is distinguishable). The major axis of one is certainly perpendicular to the principal plane of the crystal, and the major axis of the other is certainly in that plane.

In some trials of measuring the ellipticities of the rays, I seem to have arrived at the following conclusion. The proportion of the axes of the ordinary ray is more nearly one of equality than the proportion of the axes of the extraordinary

ray. For instance, with a right-handed plate of thickness 0,38 inch, and using a red glass, the first red ring was rendered ambiguous (if I may use that term to denote the state when the ring is broken in such a manner, that it is difficult to say whether the part on one side is most nearly connected with the exterior or interior part on the other side) by supplying an ordinary ray only, with elliptical light, whose

$$\frac{\text{minor axis}}{\text{major axis}} = \tan 17^{\circ}.\ 15';$$

or by supplying an extraordinary ray only with light, whose

$$\frac{\text{minor axis}}{\text{major axis}} = \tan 16^{\circ}.\ 2'.$$

With a left-handed plate of thickness 0,16 inch, the first red ring was rendered ambiguous by supplying an ordinary ray only with elliptical light, whose

$$\frac{\text{minor axis}}{\text{major axis}} = \tan 9^{\circ}.\ 3',$$

or by supplying an extraordinary ray only with light whose

$$\frac{\text{minor axis}}{\text{major axis}} = \tan 8^{\circ}.\ 50'.$$

The first result is the mean of 8 measures of each, and the second the mean of 4 measures*. The zero points of the rhomb-graduation were determined by observing when the rings of calc spar were not broken. This determination is very accurate; but if it were faulty, the effect of any error in the zero point, as well as of any imperfection in the construction of the rhomb

* In the first of these cases, (by a rough measure), the rays made with the axis of the crystal, the angle $9^{\circ}.\ 42'$ *in air*: and in the latter $13^{\circ}.\ 50'$ *in air*.

(such as I believe to exist in it) would operate in different ways with right-handed and left-handed plates: and it is on this account that I have given both. No error is to be apprehended from the jogging of the circle carrying the rhomb, as it is provided with four verniers, at 90° apart, all which were read. If this point should be made out, it may be only a consequence of the separation of the rays within the quartz: or it may be another anomaly to be added to the already sufficiently complicated phænomena of quartz. At all events, regarding the perfect equality of the ellipticities as doubtful, I have carried no farther the investigations made on that supposition: though the difference of the ellipticities is so small, that their error would be insensible.

To any one who wishes to proceed with these experiments, the following hints may be of some use.

It is convenient to have a wire carried by the rhomb, parallel or perpendicular to the plane of reflection within the rhomb, and in the place which I have mentioned as the position for a micrometer: as the observer will then see that a line from the center of the rings to the point where the ambiguity takes place, is either parallel or perpendicular to the wire.

The elliptically polarized light is of the same kind (always right-handed) when the position of the rhomb is between 0 and 90°, or between 180° and 270°: and of the same kind (always left-handed) when the position of the rhomb is between 90° and 180°, or between 270° and 360°.

The proportion of the axes is the tangent of the reading of the rhomb-position. The major axis is parallel to that edge of the end of the rhomb which makes the greatest angle with the plane of reflection at the polarizing plate.

When the ambiguity takes place near the right or left hand of the center, it is the ordinary ray only which is furnished. When it takes place nearly above or below the center, it is the extraordinary ray only which is furnished.

The following conjecture may, perhaps, without impropriety, be attached to this Paper. It is the suggestion of an explanation of the unequal refrangibility of differently coloured rays.

To account for the difference of refrangibility, we must suppose that the velocity of waves of different lengths is different either in air, or in the refracting medium, or in both. If it were different in air, it would affect the aberration of stars by a quantity that might be sensible: there is no reason to think that this is true. It is probable therefore that the difference is wholly within the refracting medium. Now it is particularly to be remarked, that the difference of velocity does not depend on the magnitude of vibration of each particle, for it is the same, whether the light be feeble or intense, that is, whether the vibration be small or great. Nor does it depend on the relative vibration of two contiguous particles, as that varies in the same proportion as the last, with a variation of the intensity. The only element which, in conjunction with either of these, will define the undulation, is the time of vibration: and it is in fact the time of vibration which distinguishes the different kinds of light. It would seem natural therefore to seek for an explanation of the difference of velocities in something which depends not on space, but on time. Now we have every reason to think that a part of the velocity of sound depends on this circumstance:

that from the suddenness of the condensation of the air, the heat evolved by that condensation has not time to escape, and the elasticity is therefore greater than if it had been slowly condensed: that, in fact, the law of elasticity is altered. Now the conjecture which I have to offer is, that perhaps there may be in refracting media something depending on time which alters their elasticity, in the same manner in which heat alters the elasticity of the air: that as in air the elasticity is greater with a quick vibration of particles than it would be if the vibration were exceedingly slow, so also in the refracting media, the elasticity may be greater with a quick vibration, than with one somewhat slower. Or perhaps the contrary effect may follow: if the vibration be quick, the latent heat (or whatever it is) may not have time to come to the exercise of its influence on the elasticity. In the latter case the elasticity, and consequently the velocity of transmission, would be greatest for the slowest vibrations (that is for the red rays) and therefore they would be the least refracted. I am not prepared to say whether the general law of superposition of small vibrations would hold on this supposition.

G. B. AIRY.

Observatory,
April 13, 1831.

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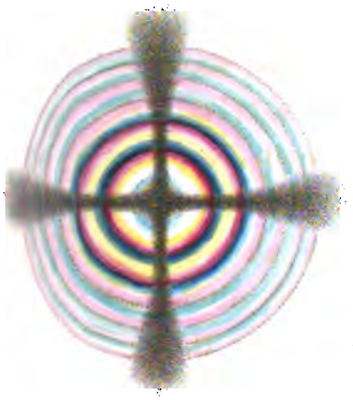

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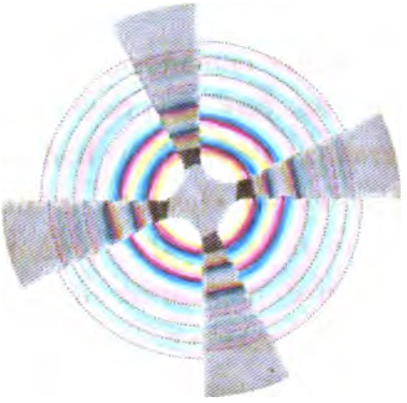

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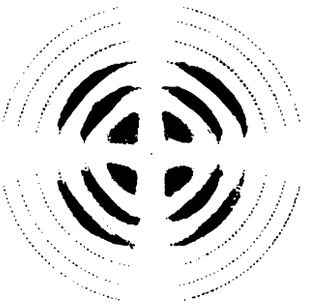

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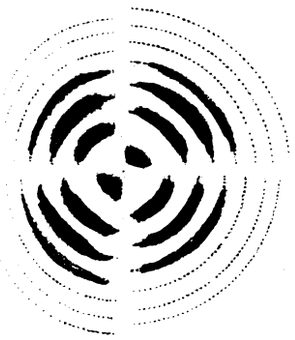

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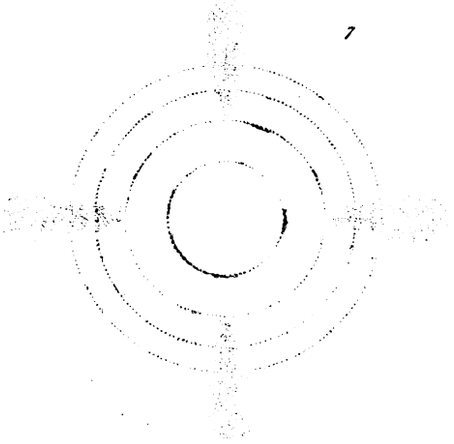

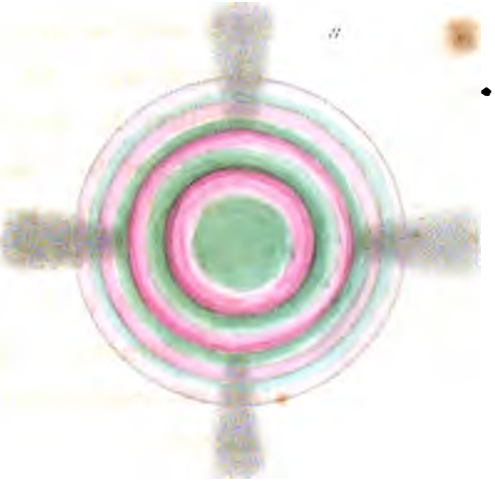

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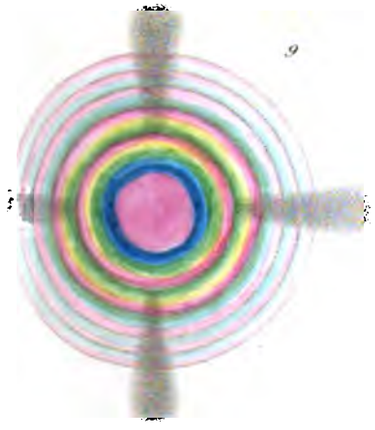

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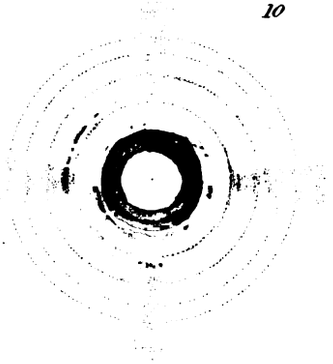

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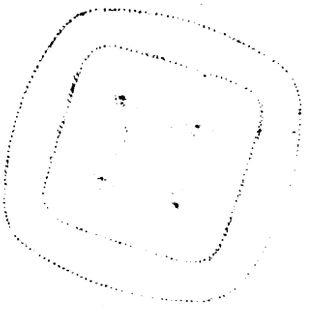

J.W. Lowry Sculp.

Fig. 12

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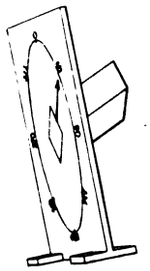

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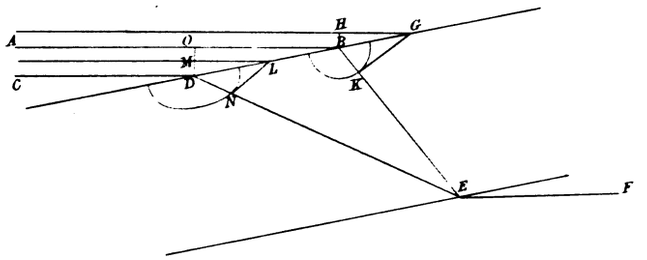

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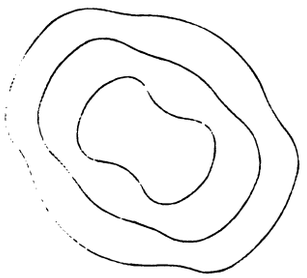

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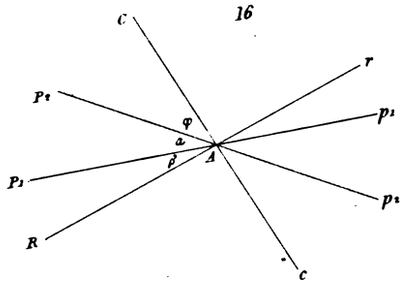

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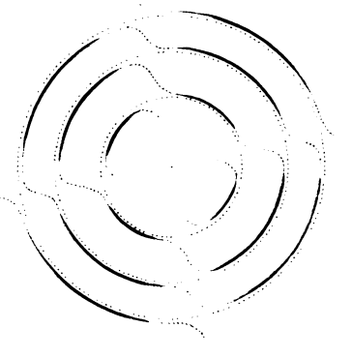

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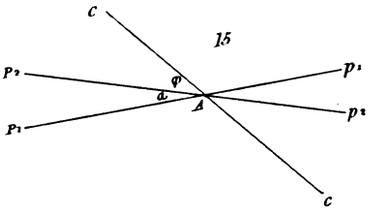

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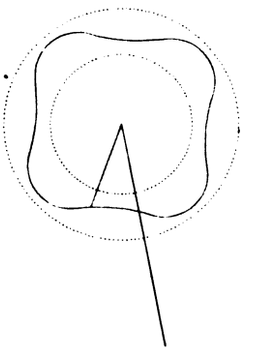

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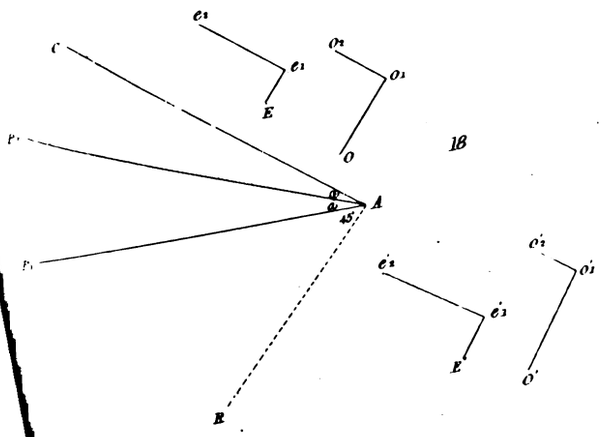

J. W. Lowry

TRANSACTIONS

OF THE

CAMBRIDGE

PHILOSOPHICAL SOCIETY.

VOL. IV. PART II.

VI. *Description of Chiasognathus Grantii, a new Lucanideous Insect forming the type of an undescribed Genus, together with some brief Remarks upon its Structure and Affinities. In a Letter addressed to one of the Secretaries.*

By J. F. STEPHENS, Esq. F.L.S.

[Read *May* 16, 1831.]

My Dear Henslow,

The magnificent Beetle submitted by you to my inspection proves to be, as I anticipated, not only perfectly novel to science as a species, but forms the type of a genus as interesting from its structure, as it is remarkable for its splendour and colouring. I shall therefore very briefly notice some of its peculiarities, though I cannot but regret that from the circumstance of my attention having been chiefly directed to indigenous entomology, the task should have devolved upon one so little conversant with exotic forms, and more especially as Great Britain is remarkably deficient in the group to which the present insect belongs.

The Lucanidæ, to which family Chiasognathus appertains, are distinguished amongst other characters by the extraordinary developement of the mandibles in the males, which in the common Stag-beetle rarely exceed half the length of the body,

but in Chiasognathus they acquire an elongation exceeding that of the body; they are extremely strong and robust at the base, and evidently capable of biting very sharply; towards the middle they become flattened, and at the tips they are incurved so as to cross over each other—whence the origin of the name I have applied to the genus*—the internal edge is irregularly serrated throughout, with a large tooth towards the base, and the apex has an acute recurved hook (Plate X. Fig. 4). In the genus Lucanus a small tubercle may be observed at the outer base of each mandible; in the insect now under examination this is greatly developed, and forms an acute spinous process about one-third the length of the mandibles, serrated within, and pointing inwards, so that *in situ* the two cross each other towards the apex similarly to the mandibles themselves.

The upper lip (*labrum*,) is very distinct, being composed of a coriaceous plate with a strong rib down the centre.

The lower jaws (*maxillæ*, Plate IX. Fig. 4.) are small, but the apical portion is very long and delicate, and fringed with very slender hairs:—the *maxillary palpi* are elongate, slender, with the basal joint very short, the second nearly as long as the others united, sub-clavate, the third half the length of the second, the terminal elongate, somewhat attenuated.

The lower lip (*labium*, Pl. IX. Fig. 8.) is membranaceous, with two, rather broad, flat, elongate, laciniæ, the extreme edge of which is finely ciliated:—the *labial palpi* are short, with the basal joint very short, the second rather longer, the terminal one nearly as long as the other two united, and attenuated.

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* Χιαζω decusso, Γναθος maxilla.

The *mentum* (Pl. IX. Figs. 5. and 7.) is transverse, semicircular, and notched anteriorly.

The *Antennæ* are remarkable for the extraordinary elongation of the basal joint and the whorl of hairs which ornament its tip; the remaining joints are short, the three first somewhat obconic, the two following transverse and produced within, the four terminal ones also produced within; the process being longer, and the articulations more distinctly laminated.

The furcate anterior portion of the head (*clypeus*), and above all, the distinct existence of four eyes, as well as the great strength of the fore legs, are characters of no little importance: to which may be added the superb colours with which nearly the entire insect is adorned; the castaneous golden-bronze of the elytra, the burnished golden-green of the gibbous centre of the thorax, and the iridescent hues of its sides and of its posterior spines, form an assemblage of intense tints rarely united into one form. In fact every part of this unique insect possesses characters of extreme interest, as may be clearly perceived through the medium of the accompanying figures, executed by my friend Mr. Westwood, who in his delineations observed several peculiarities which he kindly pointed out to me.

The food of the Lucanidæ consists of the flowing sap of decaying trees, which in the typical genus is lapped up by the four plates or laciniæ of the maxillæ and lower lip; but in this insect the very arched form of the mandibles appears to form an obstacle to the application of the laminæ to the tree unless the mandibles be opened to a great extent, as may be readily seen by the lateral view (Pl. X. Fig. 3.) given in the accompanying figure.

Respecting the affinities of this insect, the genus which makes the nearest approach to it is evidently Pholidotus, with which it

somewhat agrees in the structure of its maxillæ and of the inferior portion of the trophi, excepting that the mentum is glabrous; the genus above mentioned is manifestly allied to Lamprima, and these two genera with Chiasognathus contain the only Lucanideous insects (with the exception of Platycerus) that are adorned with metallic colours. In Lamprima the maxillæ are short; the basal joint of the antennæ shorter than the remainder taken together; the mandibles slightly elongated; but in Pholidotus the maxillæ are elongated, being furnished with a penicilliform process as in Lucanus, the three apical joints of the clava of the antennæ are alone enlarged, and the basal joint is longer than the remainder, the mandibles are large, clothed with down on their inner surface, and the mesosternum is slightly produced in front as in Lamprima. Chiasognathus therefore, by varying from the above allied genera in several of these particulars, makes a near approach to Lucanus, which is the only genus of the group containing species which may vie with it in bulk, strength of mandibles, habit, and general conformation; hence it evidently forms a truly beautiful and interesting link between the two conterminous genera Lucanus and Pholidotus, or the two families Lucanidæ and Lamprimidæ, possessing the gigantic structure of the former, and the resplendent hues of the latter family.

CHIASOGNATHUS.

Antennæ fractæ, articulo primo longissimo, sub-flexuoso, gracili, ad apicem incrassato et fasciculo pilorum instructo, clavâ pectinatâ, sex-lamellatâ, articulis quatuor ultimis sub-æqualibus.

Labrum distinctum, sub-coriaceum, carinatum.

Mandibulæ longissimæ, intus serratæ, apice incurvæ et decussatæ, basi incrassatæ, subtus processu spiniformi, intus serrato, armatæ.

Maxillæ processu apicali longissimo, exserto, gracili, sub-attenuato, pilis brevibus instructo.

Palpi maxillares elongati, graciles, articulo primo brevissimo, secundo longissimo, sub-clavato, tertio dimidio breviore, ultimo elongato, sub-attenuato.

Labium membranaceum, laciniis duabus, sub-latis, elongatis, armatum.

Palpi labiales breves, articulo primo brevi, secundo longiore, tertio longissimo, attenuato.

Mentum transversum, semicirculare, natice emarginatum.

Corpus depressiusculum, sub-latum. *Caput* latum, transversum, sub-triangulare; *clypeo* furcato emarginato, versus latera utrinque sub-spinoso. *Oculi* quatuor, convexi. *Thorax* truncato-conicus, antice in medio rotundatus et pone oculos excisus, lateribus dilatatis, deflexis, ad angulum posticum profundè emarginatis, spinis duabus acutis armatis, basi bisinuato. *Scutellum* rotundatum. *Pedes* elongati, femoribus anticis magnis, sub-cylindricis, sub-glabris, intermediis et posticis gracilioribus, tenuè pilosis, marginibus anterioribus et posterioribus densè

ciliatis; *tibiis* anticis elongatis compressis, apice emarginato, externè sub-convexis, internè sub-planis, versus apicem paulò curvatis, prope marginem anticum spinis acutis armatis, extus multidentatis, dentis duobus apicalibus magnis; intermediis et posticis brevioribus, teretibus, apice interno bicalcarato, extus serratis; *tarsi* articulis quatuor basalibus sub-æqualibus, ultimo elongato clavato, unguiculis duabus compressis, acutis, curvatis.

SP. 1. Grantii. *Sub-viridi-aureus, cupreo refulgens, thorace gibbo viridi intense, lateribus angulisque posticis versicoloribus, elytris sub-castaneo-tinctis, femoribus aureo-viridibus, tibiis cupreo-ferrugineis, tarsis antennisque nigris, abdominis segmentorum marginibus testaceis, pectore lanugine griseo tecto.* (Longitudo corporis ♂ 3 inc. 3½ lin.).

In honorem Di. Geo. Grant, M. D. hoc splendidissimum insectum nominavi.

DESCRIPTION.

Mandibles (of the male) rather more than half the length of the body, finely but distantly punctured, with a few longitudinal wrinkles at the base, the inferior process very glossy and impunctate, colour rich copper, tinted with brilliant shades of a golden hue, the base and its appendage rich blue-green and iridescent; the curvatures bronzed-black; ***head*** bright golden-green, varying in tint with the light, the disc very glossy, remotely, but finely, punctate, bluish; the lateral and posterior margins of a variable golden hue, with the spinous processes of

the clypeus purplish bronze: *eyes* glaucous: *thorax* unequal, truncate-conic, with the anterior margin rounded in the middle, and gradually excised on each side behind the eyes, the anterior lateral angle acute; the lateral margins dilated and deflexed, with a deep impunctate fovea on each, and the hinder angle with a profound circular excision terminating in two spines, of which the posterior one is longest; the base is laminated; the disc gibbous, rugose-punctate anteriorly, very glossy and impunctate posteriorly, and produced into an abbreviated ridge, composed of two lunules; the colour is intense golden-green, with the anterior and posterior margins rich purplish-copper, varying in certain positions of light to blue and violet, especially towards the posterior angles, and the foveæ on the lateral margin are fine purplish-green; *scutellum* of this last tint, impunctate, its base clothed with short pale hairs in the centre; *elytra* very finely granulated throughout, except the humeral elevation, which is smooth, of a greenish-chesnut, with a coppery or golden hue, the suture, apex, and a narrow indeterminate streak towards the lateral margin somewhat ferruginous, the lateral margin itself and the reflexed portion of the elytra bright green with iridescent tinges. Beneath: the sides of the *head* are bright green, slightly iridescent, and regulose-punctate, the *palpi* and *throat* black, the latter impunctate, the *mentum* deep violet; *breast* purplish-copper, slightly punctate on the sides, with a few transverse wrinkles between the anterior legs: *abdomen*, anteriorly, or rather *post-pectus*, densely clothed with pale lutescent hairs, the remaining portion bluish-green, slightly pilose, with the margins of the segments pale testaceous: *femora* finely pubescent above, glabrous beneath, the four hinder ones producing a dense fringe of pale hairs anteriorly and posteriorly, all of a bright bluish-

green, tinted with coppery above; *tibiæ* above coppery-green, with a tendency to castaneous, beneath rusty-chesnut, tinged in certain directions of light with greenish; the serrations, tubercular processes, and spines at the apex purplish-black; *tarsi* brown-black; *antennæ* the same; with the singular fascicle of hair at the apex of the basal joint pale griseous. At the base of the mandibles exteriorly, and on the sides of the thorax are a few very short pale scattered hairs, and on the ridge which divides the eyes is a long delicate fringe of similarly coloured ones; the anterior and posterior margins of the thorax both above and below are also densely ciliated with short pale hairs.

The female is unknown, but in all probability, when discovered, the mandibles will be found to be abbreviated, and the form of the clypeus and thorax slightly different from those of the male.

I am, Your's, &c.

J. F. STEPHENS.

P. S. Dr. GRANT, the gentleman who presented this interesting specimen to the Society, was surgeon on board H. M. S. Forte, when she returned to England in the summer of 1830, from the South American station. The Insect was brought to him in January by a native, who stated that he had found it on a resinous shrubby plant in the Island of Chiloe, which is separated from the Main Land at Valparaiso by a very narrow channel.—It appeared to have been a recent capture.

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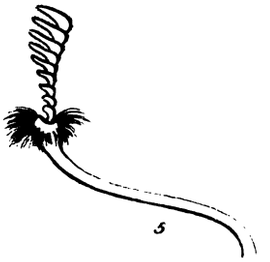

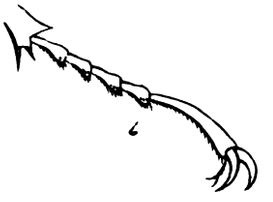

EXPLANATION OF THE PLATES.

Plate	Fig.	
IX.	1. 2.	CHIASOGNATHUS GRANTII, under side.
	3. 4.	*Maxillæ* with lacinia and palpus.
	5.	*Mentum*, processes of labium and *palpi*, under view.
	6.	Base of anterior femora.
	7.	*Mentum*, labium, &c. upper view.
	8.	*Labium* with processes and palpi, lateral view.
X.	1.	CHIASOGNATHUS GRANTII, upper view.
	2.	*Mandibles, head, thorax*, &c.
	3.	Lateral view of ditto.
	4.	Apex of mandibles.
	5.	Antennæ.
	6.	Tarsus.

VII. *A Case of Human Monstrosity, with a Commentary.*

By W. CLARK, M.D. &c. &c.

LATE FELLOW OF TRINITY COLLEGE, AND PROFESSOR OF ANATOMY IN THE UNIVERSITY OF CAMBRIDGE.

[Read *May* 16, 1831.]

Of late years no subject has more incessantly occupied the labours of learned continental Anatomists, than the investigation of the steps by which the rudimentary organs of embryos advance to their perfect form. Nor has any proved more fertile in results the most valuable. It was observed that in all vertebral animals at least, that process is effected according to a plan which is uniform for all: that in the lower orders, the full development of this plan is arrested in its progress; whilst in the higher, according to their place in the scale, it is constantly advanced to a still increasing degree of perfection. By a skilful application of this general principle it was, that Cuvier and Geoffroy St. Hilaire were so successful in explaining the osseous system in the crania of reptiles, fishes, and birds—which, up to their day, had been nearly unintelligible. And thus also, that Serres and Tiedemann, and Gall and Spurzheim, succeeded in assigning their true import to the different portions of the brain in birds and in fishes, by shewing to what parts of the rudimentary brain of the higher class they are analagous. The nature of such enquiries necessarily

drew the attention of those engaged in them, to the consideration of those unusual or imperfect forms of animals which had hitherto been considered to defy subservience to general laws, and which had been collected in a heterogeneous mass, as objects of ignorant curiosity, or as proofs that nature is sometimes capricious, and disdains an absolute submission to those forms which she herself had consecrated. An extended enquiry soon led to the conclusion that all the known aberrations from usual standards may be referred to one of three orders: according as they are characterized by defect, or by excess in the development, or by the inversion of parts. With respect to anomalies from defect, it was further ascertained that those organs in which they occur have been arrested in some one of those transient stages through which they were passing to a higher degree of perfection: and that in these stages they resemble the permanent condition of the perfect organ in some lower order of animals*. The explanation of the two other orders has not hitherto been so satisfactory. With respect to excess of development, though it has been clearly determined, that no organ in an individual of a lower class, however it may deviate from its perfect type, ever represents the type of a higher class: yet, the various instances of this mode of deviation have seemed so entirely to militate against general rules, that some distinguished anatomists have considered each case as forming a genus of itself, and as subject from its earliest period to its own peculiar laws. May it not be, however, that in the production of these unusual animal forms, the law appears to be peculiar merely because its ordinary expression or effect, has been disturbed by causes which it is very difficult to assign, and

* In a great many instances also the cause of the arrest has been assigned.

when assigned to estimate? It is with pleasure that I embrace an opportunity of proving, as I hope to do, that, in one instance at least, the apparently excessive development is referable to the usual laws.

The monstrous production which I proceed to describe, consists of the junction of two males by the head, neck, sternum, and abdomen of each, as low down as the passage for the common umbilical chord. From this point the bodies are separate. The external organs of generation are natural, the testes on the point of descending into the scrotum, the opening of the rectum, perfect in both. There are two arms, and two legs to each fœtus, and the whole body of each, from the umbilicus downward, is well nourished, and very perfectly moulded.

The connexion is of this kind. The spine of one fœtus is to the extreme right, of the other to the extreme left. The ribs from each spine arch towards a mid plane, and thus form a body with breasts on each front common to the two. The occiput of each arches from its spine, upwards, backwards, and then forwards, until it meets the corresponding parts from the other spine. And then a mixing together of the two heads takes place. The faces, in this operation are not destroyed, for the right portion of the face of one fœtus joins with the left portion of the face of the other, to make up two anomalous countenances, which look forward and backward, Janus-like, in a direction perpendicular to the common plane of the spines.

One of these faces is more perfect than the other. There are two ears to each, and a mouth: over the mouth is, in each, an eye-ball, surrounded by four eye-lids, of which the external angles are naturally united, and the internal angles meet in the mid line of the face, above and below the eye-ball. In the more perfect

face, the eye-ball presents two corneæ of different sizes: in the less perfect, one large transparent disc. Plate 11.

In the less perfect face, below the junction of the eye-brows is a very small fleshy tubercle, which is the only rudiment of a nose. In the other face this tubercle is considerably larger: is evidently bony from its resistance: is covered by a loose flap of skin: and, when this is turned down over the eye, the tubercle is found to present a cell on either side, separated by a middle septum. Plate 12. Fig. *E* and *F*.

The mouth in the more perfect face is fairly formed. When the lips are separated, a well developed tongue is seen. The mouth of the less perfect is much smaller. No tongue is here visible: but this, as will be afterwards found, is only an apparent defect. This mouth opens by an aperture behind, about a line in diameter, into another cavity: into which cavity the mouth of the more perfect face also opens.

With respect to the osseous system, I shall detail only what is peculiar and unusual: the bones of the spinal column, of the pelvis and limbs of each fœtus, and the ribs, being normal.

The bones of the head, standing upon each vertebral column, unite to form a single cavity for the common brains of the two fœtuses.

On the top of either column is placed an occipital bone to which the temporals and parietals are attached in the usual way. The parietal bones, corresponding to one column, meet those corresponding to the other column, at their anterior and superior angles which are truncated; membrane alone uniting the truncated angles, and thus forming on the vertex of the common head a large fontanelle. Between the truncated and the anterior inferior angles of the corresponding opposite parietal bones, are situated the

frontal bones: that which is above the less perfect face being much smaller than the other, and formed of one portion only, whereas the other, as is more common, is formed of two. Plate 12. Fig. *A, B, C, D.*

It is interesting to determine in what way the bones of the two crania are modified and connected so that two faces may be formed in planes parallel to that of the spines. To this we shall be led by a consideration of the common basis of the crania. Plate 13. Fig. *K* and *L.*

The occipital, basilar, and temporal bones are naturally connected on either side, and except that the squamous portions of the latter are very small, and the petrous portions very large, present nothing remarkable. The sphenoïd is connected by its body behind to the basilar bone, by its great alæ to the anterior edge of the petrose and squamous bones, and by its ingrassial processes to the frontal bones in the normal condition. And these connexions, of course, prevail here also. But from the smallness of the squamous bone, the alæ of the sphenoïd are thrown into a direction much posterior to what is their usual direction. The same disposition occurring with respect to the sphenoïd of the other fœtus, the bodies and two great alæ, on opposite sides, oppose each other; a space intervening which is nearly quadrangular. This space is occupied by the united ingrassial processes of opposite sphenoïds closely compressed, (and therefore united by ossific matter) between the right great alæ of one, and the left of the other. The great alæ and bodies are also united behind these processes from pressure; and, the bodies still maintaining their natural connexion with the basilar bones, thus surround a central space, which is oval in form, is covered by fibrous membrane, and is exactly above that common cavity into which the two mouths open.

The æthmoïd bone is entirely wanting on the side corresponding to the less perfect face, and the frontal bones are there united by ossific matter to form the roof of the orbit and the forehead of this face: the two orbits being necessarily imperfect from the absence of the æthmoïd, and therefore forming one cavity. On the side corresponding to the more perfect face there is a rudiment of the æthmoïd, which is, in fact the proboscis of this side. It appears to have been forced downwards and forwards by the lateral pressure of the frontal bones, and is connected to their inner edges by ligament.

Thus the smallness of the squamous bone, and of the ingrassial process, with the absence of the æthmoïd, renders the position of the frontal bone and of the face necessarily lateral in respect of the spine.

In the less perfect face, from the total absence of æthmoïd and nasal bones, and the narrowness of the frontal bone, the superior maxillary bones are so much compressed laterally that their palatine plates form a ridge, rather than a horizontal lamina. Hence the cavity of the mouth is deep, but very narrow; so narrow that the tongue though small cannot project into it, but lies in that common cavity which is between the two mouths, and directly below the oval opening of the sphenoïds. The other mouth, from the width of the frontal bones, &c. is large enough to receive a well-formed tongue. The membrane which lines the sides and vault of either mouth, and admits the parotid ducts, presents a small papilla in the situation of the uvula, on whose sides are seen the openings of the eustachian tubes. Behind the tongues, which are properly attached to hyoïd bones, and these to the temporal bones, is the common sac which is the bag of the pharynx. Into this the two larynges open, covered by their epiglottides; and it terminates in a single æsophagus.

The structure of the walls of the thorax may be inferred from the description of the external appearance already given. There are twelve pairs of ribs to each fœtus. The right sternal series of one fœtus, and the left of the other are received into a common sternum: whilst the left series of the first, and right of the second are similarly received into the other sternum.

The cavity of the thorax is divided into two portions by a membraneous partition proceeding from either vertebral column. In each cavity is a perfect heart in its pericardium, with a pair of lungs: and between the two hearts pass the two tracheæ; and the single œsophagus between *them*. The hearts are entirely unconnected. And the lungs have this singularity, that of the pair attached to a heart, one is the right lung of one fœtus, and the other the left lung of the other fœtus. The thorax is closed below by a large diaphragm, with four pillars; two to the lumbar region of each fœtus, with united tendinous centres. In this diaphragm there are double the number of ordinary foramina, one excepted, since there is but one œsophagus.

The œsophagus opens into a stomach (2½ inches long by 1½ in width) which rather represents a square pouch than the ordinary curvilinear form of that viscus. One extremity may be called the cardia, because there is a spleen closely attached to it. There is another spleen also, but it is attached to the stomach by a much wider peritoneal fold. From the stomach proceeds the small intestine corresponding to the duodenum; it is a wide straight intestine doubled upon itself, like the beginning of the cœcum in ruminating animals, and lying across the upper part of the common abdominal cavity. To this duodenum there passes the biliary duct from a large liver with a gall bladder belonging to one fœtus, as *A*: but it is not pervious so far as the intestine.

There is also a pancreas. The length of this portion of intestine is 9½ inches long. It opens into a sac nearly of the same dimensions and form as the stomach, situated in the common abdominal space immediately over the umbilicus.

There is another liver belonging to the fœtus *B*, with a gall bladder and ducts. The common duct is perforate for a very small space, and ends in a band of condensed cellular substance covered by peritoneum: which band is connected, by one extremity with the second spleen, and by the other with the lower portion of the duodenum already mentioned. To the sides of the sac, which filled the lower part of the common abdomen above the umbilicus, are attached two coils of small intestine, each closely packed, and each opening into, or arising from, the sac. The coil, peculiar to the fœtus *A* is 14½ inches long, that to the fœtus *B* 11 inches. They both terminate in a cœcum with an appendix vermiformis 1½ long in each. The large intestine from the cœcum to the anus, is, for the fœtus *A* 14 inches long, for *B* 15 inches. The large intestines in both have this peculiarity, that they are attached to a fan-like mesentery, and form coils like the small intestines in the natural state. Their proportional length is unusually great. Plate 13. Fig. *I.*

The urinary organs in either fœtus are altogether natural.

The common umbilical chord consists of four arteries, and two veins; a vein to each liver, and the arteries as usual.

The arteral and venous systems are similar for each heart and each fœtus.

From the left ventricle arises an aorta which, instead of the three trunks which usually spring from its arch, sends off only one: and this divides almost immediately into the common carotids for one face, or (more correctly) into the right common carotid of

one fœtus and the left of the other. These arteries present nearly the usual subdivisions, the right carotid is the largest, has a much more oblique course to the head than the other: it also gives off the greater number of branches, the arteries of the thyroid in a great measure coming from it alone. It divides as usual at the angle of the jaw. The other carotid, also sends off a thyroid, facial, internal maxillary, occipital, temporal, &c. but these are small branches, and it principally goes to the brain as internal carotid.

The aorta then arches gently downwards to the spine of the fœtus *B*. It then gives off the left subclavian to this fœtus next the right, and from these the two vertebral arteries, &c. arise. It finally terminates in the two umbilical arteries of this fœtus after supplying all the parts nearly in the usual manner.

The veins which return the blood from these arteries have a curious destination. The ascending cava enters the right auricle of the *other* heart, and the azygos and right subclavian veins join the descending cava of that heart also: whilst the veins accompanying the arteries which arise from the arch of the aorta, and the left subclavian vein form the descending cava of the same heart from whence those arteries arose.

The arterial and venous systems of the other heart are exactly similar to what has been described. So that the brain of one fœtus derives its internal carotid on one side and its two basilar arteries from one heart, and the other internal carotid from the other heart. It follows that the faces and eyes, moulded though they be from corresponding parts of different fœtuses, derive their blood, nevertheless, from the same heart.

The dura mater passing from the occiput and tentorium of one fœtus in the usual way, joins that coming similarly from the other, to form a common falx. The cerebral lobes are moulded into one mass on either side of the falx, and present no appearance

of convolutions. Neither is there any sulcus on the upper surface of that side of the brains which corresponds to the less perfect face. On the corresponding surface of the other, however, is a deep oblique fissure, as though the anterior lobes of these opposing hemispheres had not united throughout. The ventricles on either side form a common cavity.

The crura cerebri of the opposite brains are melted together over the oval aperture of the united sphenoïd bones. From this point the tubera annularia diverge separately: and are connected to well formed, distinct cerebella. The medullæ oblongatæ and spinales proceed from them naturally.

There is no vestige of any olfactory nerve. The optic nerves are very small. The right optic tractus, of one fœtus, converges to its ingrassial bone; at which point it meets the left optic tractus of the other fœtus. Here the two unite to form a *single* optic nerve for either face.

There is nothing remarkable in the other nerves. They have their usual apparent origins, and their usual passages through the cranial bones. The left vagi send a recurrent nerve round the aortas: the right do not surround any artery. They appear to waste themselves almost entirely in forming the pulmonary and cardiac plexus, and thus connect the separate sympathetic systems in the neck and thorax.

Dimensions supposing the more perfect face to look forward:

	Right Fœtus.	*Left Fœtus.*
Total length from vertex to heel	14.5 inches	13.9 inches.
From vertex to common umbilicus		6.0
Shoulder point to shoulder point	3.5	3.2
Shoulder point to end of fingers	6.0	6.0
Round the elbow	2.5	2.3
Length of foot	2.0	1.8

Circumference of common head		12 inches.
..... neck		8 .....
..... thorax		$11\frac{1}{2}$ .....
Diameter from occiput to occipal		5.0.....
..... eye to eye		4.0.....

Such being the structure of this curious production, we are naturally led to admire those deviations from the usual disposition of essential parts which seem to fit it for an independent existence after birth. The respiratory, the circulating, the nutrient, the secretory apparatus are perfect, and that it did not live, is to be attributed to accidental causes, which are unknown. Here also we have an instance of *one* creature, admirably compacted from the parts of two. For the union of the instruments of intelligence, viz. the cerebral lobes, and the nerves which supply the organs of sense, constitute it one individual. The spinal chords also are intimately united, since the corresponding crura cerebri to which they pass, form one and the same mass. The curious disposition of the lungs, and the connection of the arterial and venous system of either body with different hearts, are evidently causes which tend to the same effect. For without this, or something equivalent, the circulation which is now *one* for the whole mass, would have been *two*, a distinct and independent circulation for each body.

There is a description of a human monster, in many respects resembling the present, given by Brugnoni in the sixth Volume of the Memoirs of the Academy of Turin: and another by Duvernoy in the third Volume of the Commentaries of the Petersburgh Academy. They had, like ours, the head, neck, and upper part of the trunk, semi-double; four arms and four legs. They were both formed of the union of two females. In the Turin paper no description is given of the circulation, except that there were two

hearts, enclosed in their proper pericardia, with their veins and arteries. In the Petersburgh case, the circulation entirely differed from ours, but yet was single for the whole mass. There were two hearts, one much more perfect than the other: which supplied the greatest number of arteries, as the two aortæ, and received the greatest number of veins. The imperfect heart, after supplying a portion of the arteries of the head and lungs, and receiving some of the veins, anastomosed by its descending aorta, with the principal vessel of the other heart, and by a large venous trunk with the descending cava of the same. This case of Duvernoy, as far as the circulation is concerned, approximates to one lately described by Barkow, Chap. II. No. 6059 of the Berlin Museum: two males, with the brains nearly distinct. I know indeed of no recorded instance of a human monster with a circulation nearly similar to that which I have described. But I suspect the Turin monster to have been such. The circulation in a double pig, described by Haller, Op. Min. Vol. III. Sect. 16, very nearly resembles it. And that by Antomarchi in a double sheep, Ann. des Sc. Nat. Tom. XIV. is also nearly similar.

It has long been a question, whether double monsters arise from a single germ, or from two germs accidentally united. But the question ought to be much restricted before an answer can be given to it. Every appearance is monstrous which is unusual: and causes may exist capable of producing, in the uterus, an unnatural union of twins, more or less complete in appearance and yet not involving essential parts: capable therefore of producing a double monster. Such slight connexions as that of the Siamese youths may be conceived to have been thus produced, without doing much violence to the imagination. But to suppose that pressure so forcible should have been applied to twin germs, as to melt

them partially into one substance, without destroying their vitality: and at the same time so exactly applied that a single æsophagus is neatly compacted from the divided æsophagi of two; a single duodenum from two tortuous tubes, of which the corresponding parts were never in the same plane: that the sterna of the germs should have been exactly cloven in the mid line, and the divided parts then as dextrously united, interchangably, the half of one to the half of the other: that the aorta from one heart should, at any period of its growth, have been violently torn away, and then inserted into the other, the same operations being repeated on that other: that parts so essential to life should thus be decomposed, and re-united under another shape, seems quite impossible. It was in this revolting form that the difficulties of the question presented themselves to some of the advocates of that opinion which derives the origin of double monsters from an originally monstrous germ. But in the present day, when the successive steps of development are better, though not perfectly understood, they cannot have the same weight. On the contrary the objections are refuted by direct observation, and the suppositions on which they are built are found to be entirely erroneous.

Before endeavouring to explain the anomalies of the present case, it is necessary that I should refer to what is known of the progress of the embryo from its very earliest moments. And I undertake this task the more readily because there is no work of an English author which gives a detailed and at the same time an accurate account of the process: though very valuable detached observations may be collected from the admirable works of Harvey, Needham, Monro, Hunter, Cruickshanks and Home. It is to foreigners that we are indebted for accurate explanations of the membranes of the ovum, of the modes in which the several organs

are formed, and of their early connexions with each other. And once for all I refer below to those sufficient sources of information*.

An ovum is an organic production of a peculiar organ, the ovary, in the form of a vesicle, containing a matter which through a series of changes produces a new individual capable, when it has reached a certain stage of its existence, of supporting an independent life. Thus in all ova there are three parts, which are the product of the ovary itself, viz. a nutrient matter, a membrane to invest the whole, and an intermediate nucleus, or center, from which the formation proceeds.

The vitellus, or yolk, the earliest part of the egg, is an immediate secretion from the ovary. It is contained in the membrane of the ovum, which is smooth within and granular externally and is called the cortical membrane, or exochorion†. Between these two is gradually disposed the third part of the egg. This

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* Wolff. Uber die Bildung des Darmcanales, &c. mit Ammerkungen von J. F. Meckel Halle, 1812.

Purkinje. Symbolæ ad ovi avium historiam ante incubationem. Vratislav, 1825.

Pander. Diss. sistens Historiam Metamorphoseos quam ovum incubatum prioribus quinque diebus subit. Wirceb, 1817.

Dutrochet. Cuvier. Mem. du Museum, Vol. III. 1817.

Geoffroy St. Hilaire. Monstruosités Humaines. Par. 1822.

De Baer. De ovi Mammalium et hominum genesi. Lips. 1827.

——— Entwickelungsgeschichte der Thiere. Königsberg, 1824.

Burdach Physiologie. Leipsig, 1828.

Serres. Prevôst et Dumas. Annales des Sciences Nat. Vol. XI. XII. XVI. XXI. 1827 et subseq.

† Dutrochet applies this term to that part of the vascular membrane of the allantoïs which is outward, when that sac is doubled over the embryo and its amnion.

appears at first as an aggregation of minute globules, which arrange themselves on the inner surface of the exochorion. In some one part they are accumulated in larger quantities so as to represent a disc with a nucleus pointing towards the center of the egg. From the interior of the yolk there proceeds a minute vesicle containing a fluid. It moves towards the nucleus, penetrates it, and then bursts. The fluid diffuses itself amongst the globules of the disc, which then assumes a membranous form, and is called the cicatricula, the germinative membrane, or blastoderma of Pander. The central vesicle is supposed to be the maternal portion of the future germ, and when this process has taken place the ovum is capable of impregnation.

It was long doubted where the first appearance of the ovum in mammalia could be ascertained. The observations of Cruickshanks, of Burns, of Prevost and Dumas, had sufficiently proved that it may be perceived before its arrival in the uterus: for they discovered it in the Fallopian tubes. But, since it had escaped detection in the ovary, the ovarian vesicle of De Graaf, in the absence of any other ascertained rudiment, was considered to afford the elementary fluid from which, at least in part, it is afterwards produced. At length by the more accurate observations of Von Baer, the ovum itself was discovered within this vesicle. It first appears as a collection of minute granules on the inner surface of this vesicle, and is an aggregation of the globules of the fluid. These globules themselves are not solid, but are vesicles of a second order. Their investing portions coalesce to form a containing membrane, whilst the fluid portions become the contents of that membrane.

The ovum presents singular varieties of size, in the different orders of animals. In the dog it is in diameter not more than

$\frac{1}{20}$th of a line. The human ovum is still less than this: and, in general, the size of an ovum seems to be inversely proportional to the future evolution of the order. It may be mentioned, as bearing upon the subject of this paper, that two ova have been occasionally observed in one graffian vesicle, in the dog. The size of the ovum depends upon the yolk, whose magnitude varies according as it is intended to supply nutriment for the whole of fœtal life, or for a portion of it only, and this seems to affect the extent of the germinative membrane. For in birds the blastoderma occupies at first a small portion only of the surface of the yolk. In mammalia, on the contrary, it appears from the beginning to invest the whole. This at least is Burdach's opinion, though not Von Baer's.

Constituted as above described, the ovum of mammalia escaping from the ovary, is received into the Fallopian tube. In that organ it does not receive any new parts. It absorbs however, as it passes, the fluid of the tubes, and so increases in size. The great changes which it undergoes are afterwards effected in the uretus.

In the different classes of animals the successive changes in the ova are not effected in corresponding parts of their productive apparatus. In the lower forms of life the changes of the egg are begun and completed in the ovary. In invertebral animals the egg is so far completed in the ovary, that it receives only its external shell in the oviduct. In oviparous vertebral animals, as in mammalia, the yolk and the two membranes are alone formed in the ovary. Their egg however receives in the oviduct the successive deposits of albumen with its membranes, the membrane of the shell, and the shell itself: whilst in mammalia the parts analogous to the membrane of the shell, and the shell, are completed in the uterus. Thus according as the place of the animal

is higher in the scale, so does the addition of the accessory parts of its ovum occur in situations more remote from the ovary.

As the human ovum passes along the tube, the germinative membrane is more and more developed: and the activity of the uterus is at the same time exalted. This organ secretes from its entire surface a soft matter, which is spongy, reticulated, filamentous, about the tenth of an inch in thickness. The secretion is called by Hunter the decidua. It represents, of course, the exact shape of the uterus, of which it is a cast, and closes all the openings into that organ. Its attachment to the uterus is in the early stages very slight, being here and there connected to its walls by a few very delicate blood vessels. It at first merely adheres to the uterus, as a secretion, and afterwards becomes united to it by vessels. The ovum passing from the tube which opens obliquely into the uterus, meets the obstruction of the decidua: this, from its slight adhesion to the walls of the uterus, it is able to detach for a greater or less space, and then sinks, by its weight, into the body of that viscus. Thus as the ovum descends to the more central region of the uterus, it carries before it a duplicature of the decidua which supports it and keeps it in a near connexion with the uterus. The doubled portion is called the decidua reflexa. From this description it follows that at the place where the decidua is reflected, the uterus is deprived of its covering. A new secretion from the uterus shortly supplies the deficiency, and this is the seat of the future placenta. The human ovum passes into the uterus at the end of the second week after impregnation. The reflection of the decidua begins at the end of the third. The reflected portion receives a considerable supply of vessels from the uterus, and soon becomes the thicker of the two. In process of time, however, from constant distension by the gradu-

ally increasing ovum, it becomes thinner—is brought in contact with its external portion (the original decidua)—and finally, about the fourth month of pregnancy, coalesces with it.

At an early period after the egg has passed into the uterus, the granular cortical membrane expands—its granules assume an arboresent filamentous form, and thus, penetrating the reticulations of the reflected, fix it to the proper decidua and so to the uterus. Since the ovum increases considerably in size, previous to this attachment, the earliest nutrition of the embryo, is to be attributed to absorption from the fluids of the tube and of the uterus, by the granular exochorion. The series of changes by which it is perfected takes place in the blastoderma. These have not been very accurately ascertained, in the earliest stage, in the ova of mammalia: but there is ever reason to believe that they are exactly analogous to those which have been determined with much care in the hen's egg.

The first change is the gradual separation of the blastoderma in the region of the disc into what now appear to be its component parts, whilst at the same time it recedes a little from the yolk. It presents an external serous layer, and an internal mucous layer. At the same time the proportional distribution of these membranes is different in different parts of the disc: thus in the center the serous portion prevails more than the mucous, and in the circumference the mucous portion more than the serous. Hence the disc presents a pellucid area, surrounded by a broad darker ring. Between these two laminæ, a third portion of the blastoderma gradually becomes evident. It appears as a layer of granular matter: and since this matter is the substance from which the blood vessels and their contents are formed, it is called by Pander the vascular layer. There are thus three parts of the

germinative membrane essentially different: and to each is appropriated the office of laying the groundwork of distinct portions of the embryo, and to a certain point, of perfecting these. But no part of the embryo is finally completed without at least one of the other parts afterwards participating in the work.

The first perceptible indication of the embryo is a darker line in the middle of the transparent area. It is situated on the inferior surface of the serous membrane, and indicates the future spine of the animal. On this same membrane, on either side of the primitive streak, the dorsal plates are disposed, which meet in such a way as gradually to form a cavity for the future spinal cord and brain; then, on either side of these, are the ventral plates. As the process advances, the ventral plates meet in front only, and the head is bent downwards towards the yolk. So that at this stage of the process, the rudiments of the whole motive and sentient systems, are formed on the serous portion of the blastoderma; whilst its mucous portion, lying below the serous, is the open abdominal space, and rests immediately on the yolk.

In the mean time the granules of the intervening vascular layer are more developed. They are collected together more closely in the dark border, and in smaller streaks near the pellucid area. As they come in mutual contact, a change takes place in their component parts. Their external portions coalesce to form canals which include the more fluid parts: thus, gradually, are formed on the vascular membrane a number of small veins which anastamose and separate to form a fine net work, and then finally open into the large external circular vein formed similarly in the dark border, and called the 'terminal vein.' The circle is not completed at that part which is above the anterior extremity of the embryo, but either limb of the arc terminates

in a vessel which bends downwards and inwards to join the heart.

The formation of the heart is absolutely similar to that of these vessels of the areola, and takes place before the fluids in the latter can be seen to move. By the bending of the ends of the dorsal laminæ, the blastoderma is drawn down at an angle from its general direction upon the yolk, and forms a fold under the head of the embryo, in which fold its serous and mucous portions are considerably separated. This is a most important movement, for a triangular cavity is thus left between them, which is soon occupied by a collection of globules from the vascular intervening membrane: and it is from this large collection of globules that the heart is formed, by a separation of their component parts. They become, in the same way as the vessels of the areola, a longitudinal sac, with two posterior angles, containing a fluid mass. The angles receive the descending vessels from the terminal vein of the areola. At the very time that the heart is thus formed, the fluid contents of the canal for the spinal cord begin to assume a degree of solidity. The brain now consists of three vesicles containing a fluid, which are the rudiments of the medulla oblongata, of the corpora quadrigemina, and of the cerebrum. They lie, in the order mentioned, in a curved line continued from the spinal cord: and, with other parts afterwards superadded, may be observed before any vestige of a blood vessel can be detected near them. Thus the formations of the heart, and of the brain are synchronous, and have no connection as cause and effect.

It has been said that as the dorsal laminæ are bent the germinative membrane forms an angle, where it is drawn down by the head from the general direction of that membrane. This operation does not occur at the head alone, the same takes place

at the whole circuit of the embryo, which now resembles an inverted boat: so that the serous portion of the germinative membrane seems to project over the dorsal surface of the embryo, in all directions, forming a duplicature which surrounds an oval space. The laminæ of the duplicature continually grow, and advance towards the centre of this space. This point they soon attain, and then coalesce; and thus is the continuous membrane of the amnion formed: a serous sac which contains the embryo, and which is incomplete only at that part where the ventral plates have not yet coalesced, and which is the navel in mammalia, and the point of communication between the yolk and the abdominal cavity in other creatures.

The mucous portion of the blastoderma covers the interior, or abdominal surface of the embryo next its spine, this disposition being a necessary consequence of the early development of its body on the inferior surface of the serous portion. From this region it spreads itself over the yolk, between it, and the continually spreading vascular layer. The yolk thus covered, by mucous and vascular membranes, forms the umbilical vesicle of mammalia. It is at this moment in the place of the intestines. As the contents of the bag become diminished by absorption on the part of the embryo, cylindrical portions of it are folded in by the ventral bands coalescing from before backwards, and from behind forwards, the process first beginning towards the head. These cylindrical portions thus form blind diverticula, which adhere closely to the spine at the ends next the head and tail, and more loosely in the intermediate region, where they open into the umbilical vesicle. As more and more of the vesicle is drawn in and enclosed, the length of the upper and lower bowel encreases: and the ventral plates closing more and more, the passage between

the vesicle and bowel is gradually diminished, and the vesicle projects in front of the abdomen like an hernial sac. The passage from the vesicle into the bowel, or ductus vitello-intestinalis, is at that part which nearly bisects the length of the future small intestine. In man the duct becomes gradually capillary, at length ordinarily disappears, and then the only connection between the vesicle and the embryo is by means of blood vessels hereafter to be described.

Hence it appears that the groundwork of the organs of animal life are formed exclusively on the serous portion of the germinative membrane, that the system of organic life is derived originally from the mucous portion, and that the granules of the intermediate nascular layer exclusively supply the matter from which the blood vessels and their contents originate. It will be seen that these separate portions of the blastoderma afterwards combine to form organs upon which depends the advancing evolution of the embryo.

From the anterior portion of the canal of the heart, arise two vessels, these proceed forwards, surround the anterior extremity of the digestive canal, attain the region of the spine, and then unite to form the descending aorta. The aorta loses itself in a small branch on the lower portion of the spine, whilst what may now be more properly considered as its principal continuation leaves the spine at a right angle, reaches the vascular area on the yolk, ramifies there and inosculates with the veins of that membrane and finally enters the terminal vein of the vascular area. This is the first circulation: the aorta cannot be yet seen to ramify within the body of the embryo, and a considerable part of that body has as yet no principal trunk distributed to its substance. The artery which thus proceeds to the umbilical vesicle is the omphalo-mesenteric artery.

The heart by its increase soon appears to be too long for the cavity in the neck that contains it, it becomes bent upon itself. The mid portion of the canal forms a very acute angle with the posterior portion which received the veins. The angle is the future apex of the heart, whilst that part which receives the veins, afterwards becomes the sac of the auricles. There is a constriction also near its anterior extremity which denotes the bulb of the aorta.

The two vessels which have been described as arising from the bulb of the aorta, are not the only vessels which come from it. And here it is necessary to record one of the most striking anatomical dicoveries of modern times, which was made by Rathkè about the year 1823. He found that not only in fishes do branchial arches and cavities exist, but that these are found during some portion of fœtal life, as in the batrachia, so also in serpents, lizards, birds, and mammalia including man. His observations have been confirmed and illustrated by Burdach and Von Baer. In all these animals five pairs of branchial arteries arise from the bulb of the aorta: they do not all exist at one time, but appear in succession from before backward. They all arch round the beginning of the digestive canal, (the anterior approximating as nearly as possible to the rudiment of the brain,) and coalesce in the region of the spine on either side to form a trunk, which is a root of the aorta: for the two trunks unite to form that vessel. These arches are separated by deep fissures in the sides of the neck, which penetrate even into the cavity of the intestine. There are four pairs of fissures between the corresponding arches, exclusive of one in front of the most anterior pair. The four posterior fissures on each side are gradually obliterated, the anterior first disappearing, and the others in

succession. The pair of fissures anterior to the first branchial arch afterwards becomes the mouth, whilst the bands which are bounded by the first and second fissures afterwards becomes the lower jaw. In man the branchial arches appear in the fifth or sixth week, and exist only for a very short time. And here, as in the other mammalia, in birds and in the higher amphibia, they undergo no further development. In fishes, as is well known, they are respiratory organs which serve for the whole life of the individual, and are gradually endowed with an apparatus of cartilage and bone, admitting of numerous subdivisions on which the ultimate branches of the vessels are spread. In batrachia, the vessels are subdivided on membraneous productions of the soft branchial plates, and exist only during the larva state, with the exception of the Proteus and Siren where they are permanent.

With respect to the branchial arteries from the aorta, some disappear as far as their stems are concerned, whilst their ramifications persist, and others persist entirely under certain modifications. The following is the process in the chick where, for obvious reasons, it has been most accurately observed. From the simple cavity of the heart, at its anterior extremity which is the bulb of the aorta, spring in succession five pairs of branchial arteries. They arch round the beginning of the digestive cavity, to attain the spine, and on either side of it coalesce to form a single stem. The stems are the two roots of the aorta, which unite lower down in the spine to form that vessel. The trunks of the two first pairs soon disappear; whilst the arteries which orignally sprung from them to supply the head are permanent, as well as their posterior branches of communication with the subsequent pair. When this occurs, the blood flows from the heart, through the third, fourth and fifth pairs, to the roots of

the aorta, and by the communicating branch on either side, fills the arteries of the head. This third pair becomes the artery for the neck and arm, and its communication with the bulb of the aorta disappears also: the stream from the aorta then flowing in the fourth and fifth pairs from the heart. The next that disappears is the fifth branchial artery on the left side. Whilst this process is taking place, the septum of the heart is forming, from the apex upwards: the last part of the septum which is completed being that immediately under the bulb of the aorta. It is a consequence of this formation, that as the heart contracts it sends the blood into the bulb, in two separate currents. These currents surround each other in a spiral form as they advance into the aorta, which arises indifferently from the two cavities of the heart. The current from the right cavity, fills the fourth branchial artery of the left side, and the fifth of the right side, and these afterwards become the pulmonary arteries, with the exception of small branches of communication with the root of the aorta, which are the two ductus arteriosi of birds. From the left side of the heart, the current of blood is directed to the fourth branchial artery of the right side, which becomes the descending aorta. This metamorphosis prevails in mammalia, though with modifications which are not so well understood. In the second month of the human embryo according to Burdach, there are found two principal arterial stems, an ascending stem to the head, a descending stem to the body. Of these, the first appears to be the root of the third pair of branchial arteries, with the subdivisions of those anterior to it, whose roots have disappeared. Its blood comes from the left cavity of the heart. The descending branch of the aorta comes from the right cavity, and is composed of the remains of fourth and

fifth branchial pairs. The anterior and posterior aortas are still connected by a branch of communication, the remains of the original root of the aorta. When the lungs are formed, the beginning of the descending aorta supplies them with blood. Its contents are in fact entirely distributed to them, with the exception of what still passes through one remaining branch of communication with the root of the aorta on the left side, which is the ductus arteriosus, so that now the descending aorta can only receive its blood from its branch of communication with the anterior aorta. Hence the branch of communication increases, and forms between the anterior and posterior principal trunks the arch of the descending aorta. This now derives its blood entirely from the left side of the heart, and becomes the common origin of the ascending and descending vessels; whilst that which was its original stem, and which arose from the right side, now becomes the pulmonary artery.

Thus the aorta is formed from the concourse of branchial arteries which spring from the heart, and whose direction seems to be determined by the situation of the central nervous masses; for they stretch towards the rudiments of the brain and then descend in the neighbourhood of the spinal cord.

From this point the completion of the embryo requires the co-operation of at least two of the original elements of the germinative membrane.

The liver is formed by two productions of the upper part of the intestinal tube. They proceed from the intestine as cylindrical prolongations of the mucous membrane surrounded by their vascular layer; these subdivide, and form many arborescent hollow filaments. Thus are constituted two principal lobes, which surround the ascending vein to the heart. This vein sends off

a loop, which ramifies with the subdivisions of the tube, and thus the groundwork of the liver is formed, to which the parenchyme is gradually super-added. The ramifying vein is the Vena Portæ: the original venous stem, as it advances to the heart, receives the collection of returning veins from the liver, and is the center to which the ascending cava is afterwards directed.

Before the first circulation through the vascular area is fully established, the heart does not pulsate, but has only a vermicular motion from one end to the other. The blood in the aorta undulates, as in the dorsal vessel of insects, and has a different direction in different parts of the tube. In this way it seems gradually to overcome the obstructions to its direct course, and the first circulation is established. Numerous veins of the vascular area soon collect around the subdivisions of the omphalo-mesenteric artery: they form a trunk which returns to the body, in company with that artery. They separate on reaching the intestine, and the vein turns upwards, to become a principal root of the Vena Portæ. This omphalo-mesenteric vein is considered by many as the first vessel which ramifies within the body of the embryo.

The first circulation having been established, the arteries subdivide in the soft substance of the body. For instance—the contents of the posterior extremity of the aorta have an undulating motion until the artery returns upon itself and forms a loop: the advancing part of the loop is an artery, the returning part a vein. From the convexity of the loop another loop is gradually formed, and then another. So that one side of the chain of loops at length forms a continuous arterial, and the other a continuous venal stem; and the transverse portions of the loops are anastomosing branches. It seems that the smaller veins are

attracted by the larger ones, and when they come into their neighbourhood coalesce with them. Hence the veins which accompany the subdivisions of the anterior aorta, and the intercostal arteries are guided by these vessels directly to the venous sac of the heart, whilst the same sac of the heart is the centre of the veins of the abdomen and lower parts of the body. The difference is that these two series of veins attain the heart most directly, the one by following the course of the arteries, the other by diverging in some degree from that course.

But to return to the membranes of the ovum, which hitherto we have described as an external exochorion, and an internal serous amnion.

The allantoïs is a production from the posterior part of the bowel at its lower extremity. It proceeds from the cloaca, (for as yet there is no urinary bladder,) as a thin transparent whitish vesicle, through the unclosed abdomen at the part nearest to the tail, whilst the umbilical vesicle passes out nearer to the head. It consists of two layers of the blastoderma, the mucous and vascular layers. Its existence is later than that of the liver, but earlier than the production of the arteries which form the umbilical cord. It is observed in the human embryo in the third or fourth week, but here it does not attain any great size, and disappears in the sixth: whilst in many other mammalia, as also in birds, it is reflected over the whole amnion, which with the embryo it includes as a sac doubled on itself, the sac being somewhat distended with fluid. When the urinary organs are developed, the bladder is formed out of that portion of the allantoïs within the abdomen which is connected with the intestine. This at first does not increase in size with the rest of the sac, but appears rather as a duct connecting the bowel and the ex-

panded outer part of the allantoïs. As it expands, the terminating branches of the aorta, follow the vascular external layer, and run upon its sides, and so pass out of the abdomen. In birds, and the higher amphibia, the umbilical arteries ramify most minutely on the external vascular layer, particularly on that portion of the doubled bladder which is next the exochorion. In man the allantoïs external to the abdomen soon dwindles to a capillary tube, its original vascular layer still continuing as a conductor of the umbilical arteries. In the other mammalia the allantoïs persists throughout fœtal life, and receives numerous branches from the umbilical arteries. The veins which return the blood are collected into one stem which accompanies the arteries as far as the navel. There the two orders of vessels separate. The veins pass as one large trunk to the general venous center, the ascending cava, below the heart. This is the principal stem: subordinate branches are given to the liver as it passes between its great lobes, and a branch of communication to the Vena Portæ.

In man the umbilical arteries are immediately surrounded by the vascular membrane which originally belonged to the allantoïs. When the cord reaches the placenta, now beginning to be formed, that membrane spreads itself over the whole inner surface of the exochorion, and forms its vascular portion. Thus is the chorion formed; it consists of an external villous exochorion, and an internal vascular endochorion. In the case of twins the two embryos have usually a common exochorion, with the other membranes proper to each. This is an instance of the law which has been well elucidated by M. Serres, viz. that in the primitive state of embryos when two homogeneous organs are brought in contact, they coalesce and form one. If the amnion be also common to the two, the case, though not uncommon, is pathological, or monstrous.

The human placenta is developed when the umbilical vesicle begins to disappear: and what it is that determines the part of the uterus on which it is formed has been already described. The subdivisions of the umbilical vessels distribute themselves essentially to those tufts of the exochorion which are in the region where the decidua was reflected. The tufts gradually grow together in the third and fourth months of pregnancy, and thus the fœtal placenta is smooth on its inner surface and presents numerous inequalities on the external: which inequalities are the several collections of the prolonged and subdivided villi of the exochorion. On every one of the villi as a stem, and on all their subdivisions as branches, are distributed corresponding subdivisions of the umbilical vessels which directly anastomose.

As the abdomen gradually closes the only opening left is the navel, which is near the tail end. The umbilical cord consists externally of a layer from the amnion, which passes from the cuticle of the fœtus: beneath this is the vascular sheath originally derived from the allantoïs, which on the one hand is lost in the aponeurosis of the abdominal muscles, on the other in the endochorion. It immediately surrounds the vessels, as well as the layer of cellular membrane which they derive from the peritoneum, and which with the vessels contains also a gelatinous fluid, gradually to be absorbed by the fœtus, and called the gelatine of Wharton. Thus the original external membrane of the ovum, the exochorion, has at no time any direct connection with the cord, or with the embryo*.

* Besides these parts the cord at the earliest period of its formation, when it is wide towards the abdomen, contains a loop of intestine, the umbilical vesicle, and the allantoïs.

The lungs in all vertebral animals are in their origin probably processes from the mucous layer of the abdominal canal, with corresponding accompanying portions of the vascular layer, and are in their earliest stage analagous to the swimbladder of fishes, and the lungs of the Proteus and Siren. They are begun in the sixth week; and receive their further development in proportion as the other respiratory apparatus disappear. These latter are the branchial arches, and endochorion of the allantoïs with the placenta. These three last respiratory organs do not appear to be intended to perform their functions perfectly at the same time in any class of animals. When the branchial arches of birds and of the mammalia disappear, then the allantoïs is the active respiratory organ: and where the arches are permanent, as in fishes, for the whole of life, and in frogs until the lungs are developed, the allantoïs is not formed at all.

It is now time that I should declare my opinion respecting the mode in which the appearances, which the subject of this paper presents, have been produced. And this I do with much diffidence, though it has not been hastily, nor carelessly formed. I look upon it, then, as a case of twins, whose germs were originally perfect and distinct, notwithstanding all the difficulties which that opinion includes. The two ova I suppose to have been brought into close contact in the Fallopian tube, and that thus their exochorions have coalesced, as is usual, to form a common cortical membrane: and that, besides this, a coalition has also been effected between corresponding points of the germinative membrane, near the disc. The nearly perfect symmetry of the two faces, and of

the whole united portions of the two bodies, shews that if such an union did take place it must have been in corresponding parts. These parts I conclude to have been in front of the primitive streaks of the embryos. Let us now suppose that the evolution of each embryo advances, that the dorsal plates and spinal cavities and vesicles of the brain are formed in each: until the space between the anterior extremity of each is occupied by the anterior vesicle of the brain of each. These, according to M. Serres' law, will now coalesce, and our dissection shews that these are the only parts which did coalesce, for the cerebella and corpora quadrigemina were separate and distinct. The ventral plates of the opposite embryos necessarily meet in front: and there is an effort to bend the head downwards to form the cavity for the heart. This bending takes place in the direction of the ventral plates, the effort being resisted in every other. Thus there are cavities for the hearts properly formed: and they receive their blood from opposite sides of the united vascular areas. Let us now consider how the aortas are to be formed. Five pairs of branchial arches project from each heart on either side of the neck: They arch round to the basis of the brain to reach the situation of the spinal cord, and there coalesce to form the roots of the descending aorta. Thus the descending aorta will, in this case, be formed by branchial arteries which come from different hearts: they are the root of the third pairs, which arches from either heart in a direction *from right to left, until it gains the spine.* Thus all the arteries which arise below the original third branchial artery, come from the descending aorta, whilst the arteries of the head, come from the arch which connects the descending aorta to the heart. And that distribution is exactly the one before us: the aorta has reached the spine to which it does not seem to belong,

because it is formed according to the usual laws. The veins also follow the usual law;—the superior cava is guided by the arteries (vid. page 28) of the arch of the aorta to the venous side of the heart; and the inferior veins—(those which return the blood of the descending aorta of the *other* heart)—to the same cavity. It must be recollected that this venous center is determined before there are any limbs and before the aorta has subdivided in the body, and that it is quite independent of the latter vessel. There is one apparent difficulty: each fœtus receives its right subclavian artery from one heart, and returns its right subclavian vein to the other: whereas the left subclavian artery and vein are both attached to the same heart. But this also comes under a general rule when it is remembered that the subclavian veins join the *jugulars* to form a descending cava, and that the course of the jugulars is determined by the carotids, whilst the subclavian arteries are originally from the extreme upper part of the descending aorta.

The sac, by whose intervention the bowels of the two bodies are united, appears to me to be the most interesting feature of this case. I consider it to have been developed from the umbilical vesicles of the two embryos, which, according to my hypothesis, were united at their anterior parts: I believe it, in fact, to be the conjoined vesicles. It will be remembered that the vesicle, in the normal condition, at first fills the abdomen: that as the bowel is formed at its expense, it gradually retreats into the cord, being borne onward by the continually increasing loop of intestine, and that at length the duct of communication between the bowel and vesicle vanishes, and the vesicle itself rests at the farther extremity of the cord, or between the chorion and amnion in front of the placenta. In the united bodies the circumstances

are entirely different: the vesicle has four points of attachment instead of two, it is pressed by equal forces in opposite directions, and is kept at rest. Hence the upper coils of small intestine were found much compressed with a very short mesentery, and closely connected to the walls of the vesicle: whilst the lower bowels were disproportionately developed, and the mesentery much extended between them and the spinal columns. If this explanation be accepted it sets the question at rest concerning the existence of an open duct of communication between the umbilical vesicle and the small intestines in the human embryo, a communication which has been denied by great authorities, by Emmert, Hoechstetter, and Cuvier. In a rare case, (Tiedemann, Anatomie der kopflösen Missgeburten, tab. IV.), there was found a true umbilical vesicle attached to the intestinal canal. The fœtus had arrived at the full term, and was in many respects monstrous. In Brugnoni's case it is very remarkable that the single duodenum terminated in a vesicle, whilst the lower portion of the small intestines of each fœtus were not connected with it, but commenced as blind tubes.

The upper part of the intestine, that from which the œsophagus, stomach, and anterior portions of the small intestine are afterwards formed, was perhaps probably at its origin, and from the first, a single tube connected to the spine of each embryo. That it was formed from the parts appropriated to each I conclude to be a fact: because I observe that there are two pairs of lungs with a trachea for each, and two livers. The appearance also of the stomach and the presence of two spleens seems to indicate that it has been formed from the rudiments of what ought to have been two. The existence and the magnitude of the liver on the left side of the drawing Pl. 14. shews that the unintelli-

gible mass between it and the spleen was once intended for duodenum, though never matured into that viscus.

I admit that there are difficulties in the explanation which I have thus given; but I advance it because it is only by a careful consideration of individual cases that the mystery of these unusual productions can be unravelled; and what is yet obscure in organic development be satisfactorily explained. Such productions as that which I have described were formerly kept in museums as objects of idle wonder. They are now closely investigated as particular cases of a general problem: cases in which some of the conditions are modified or suppressed, and of which the value and force are therefore rendered apparent. And thus do they present ready to our hand, means for the solution of what is most difficult—devices which no human ingenuity could have contrived.

DESCRIPTION OF THE PLATES,

which are accurate orthographic projections of the original, drawn with Mr. WILLIS' Machine.

PLATE 11

Contains an anterior and a posterior view of the subject, half the real size.

PLATE 12.

A. Front view of the cranium and face, where is a rudiment of the æthmoïd bone over the orbit. Here there are two frontal bones. Between the upper and lower jaws the tongue is seen. The bones, which make up the orbit in this face, are lettered in the plan *G*.

B. Cranium and face on the opposite front, where the frontal bone is single. The bones which make up the orbit are lettered in Fig. *H*.

C. View of the vertex.

D. View of the occiput, face *A* to the right hand, face *B* to the left hand.

E. The proboscis with its skin projecting above the orbit, where the soft parts have shrunk.

F. The bone of the proboscis where the skin of the same is turned down upon the orbit.

G.
- *a.* Frontal.
- *b.* Sphenoïd.
- *c.* Maxillary.
- *d.* Malar.

H. Letters correspond to those of *G*: the orbitar frontal plate single.

m. View from above of a pair of united Ingrassial processes of opposite sphenoïd bones, under (*m*) passes the optic nerve. There is an open fissure in the mid line.

n. View of the same from below.

o. Lateral view of the same.

Plate 13.

I. View of the intestines.

K. View of the common basis of the skull.

L. Map of the same. The portions of the occipital bones are all *yellow:* of the temporal bones *red:* of the frontal bones *brown:* of the sphenoïd bone, with four great alæ, and united ingrassial processes between two of those alæ on each side, *blue.*

Plate 14.

The Circulating System.

M. Represents the bodies drawn asunder as far as possible, after the diaphragm has been removed and the hearts separated. The bodies are in the plane of the paper, the head and neck below that plane. *a a*, the inferior cava broken off near the point of its entrance into the heart, as is the azygos of this body—the Aorta of this body is from the other heart, &c. vid. page 9. The Œsophagus, and the trachea of the lungs attached to the lower heart, are seen passing in the space between the two hearts.

N. Shews the Arteries from the arch of the Aorta on one side—the veins, which form the descending cava—portions of the ascending cavæ, and of the descending aortæ. The two hearts are seen, and portions of the lung between them. This is the same view as would be presented if the upper heart in Fig. *M*, were turned down, to its proper position, and the head and neck raised

W. Clark, ad nat. dimid. mag. del.
G. Scharf, lithog.

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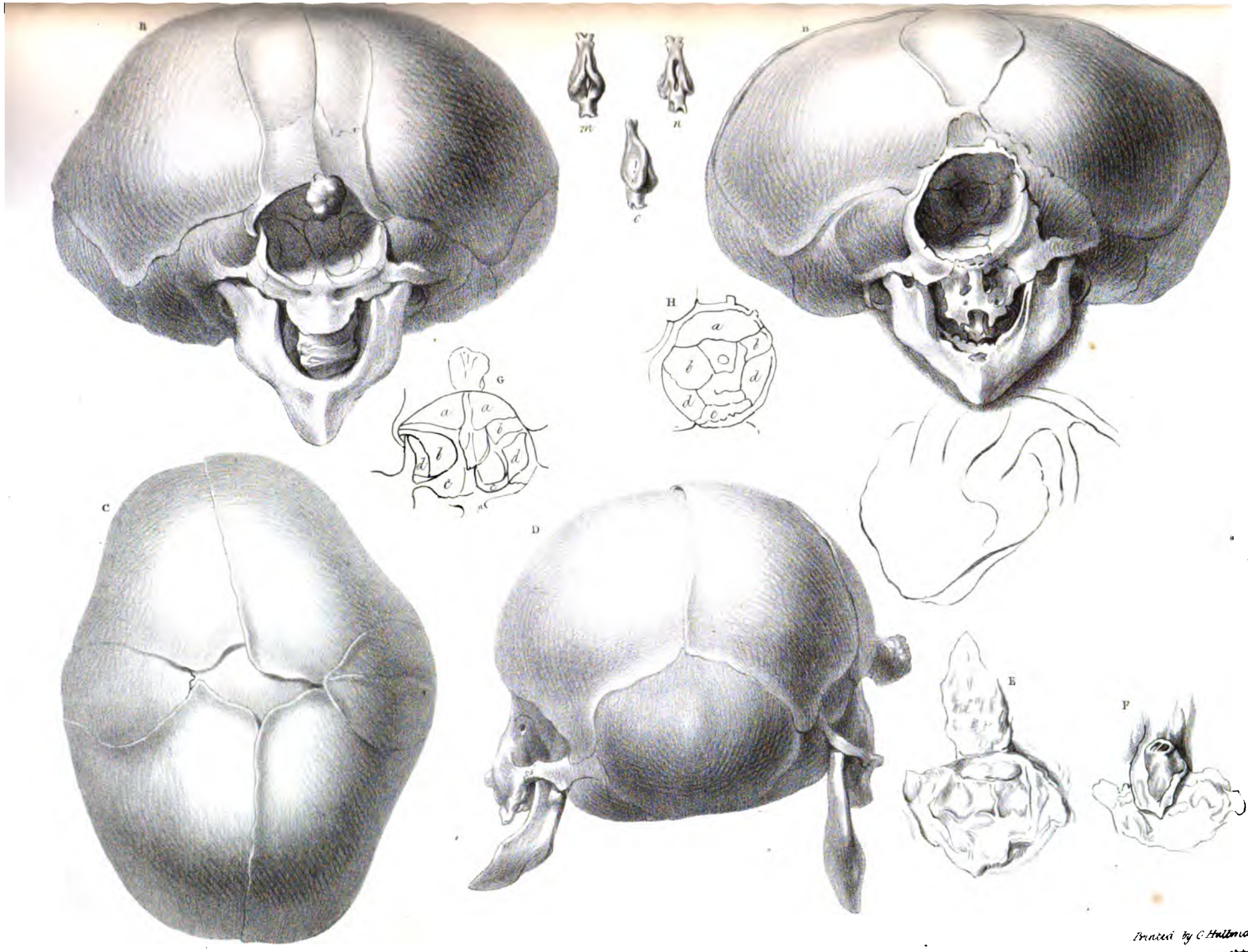

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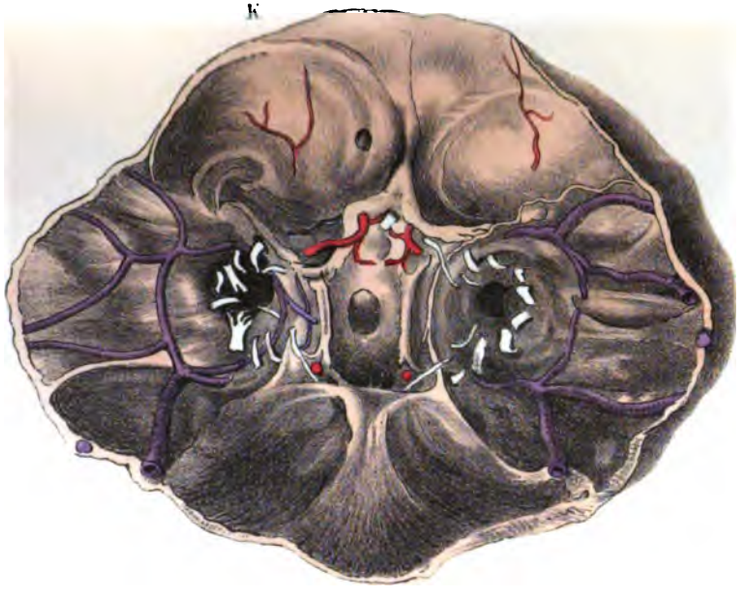

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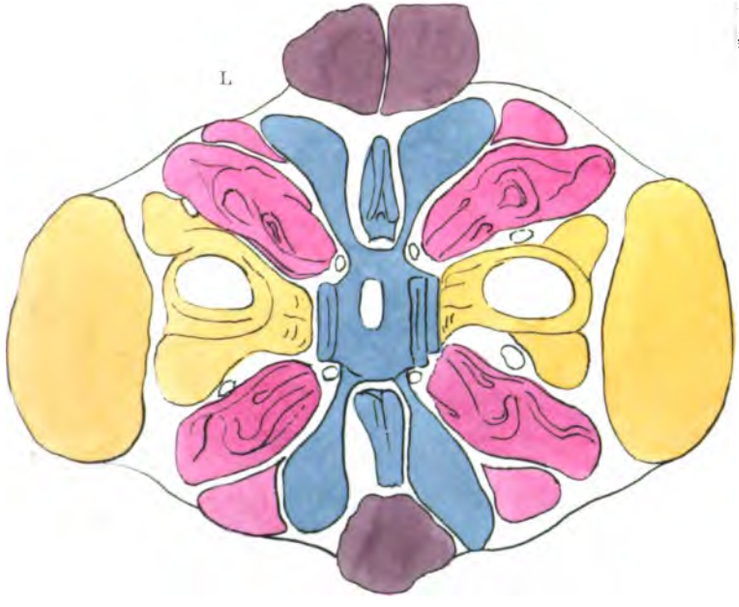

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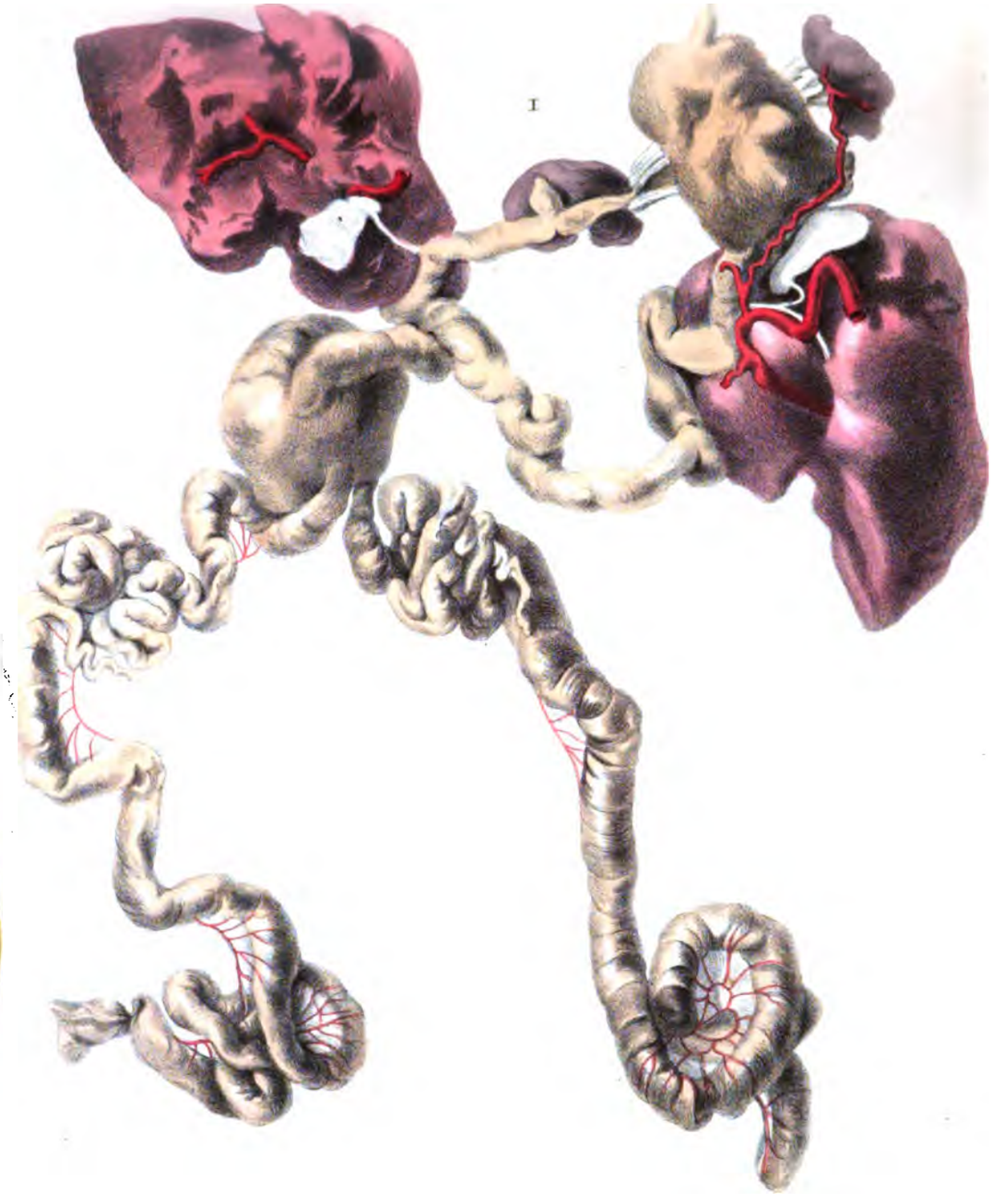

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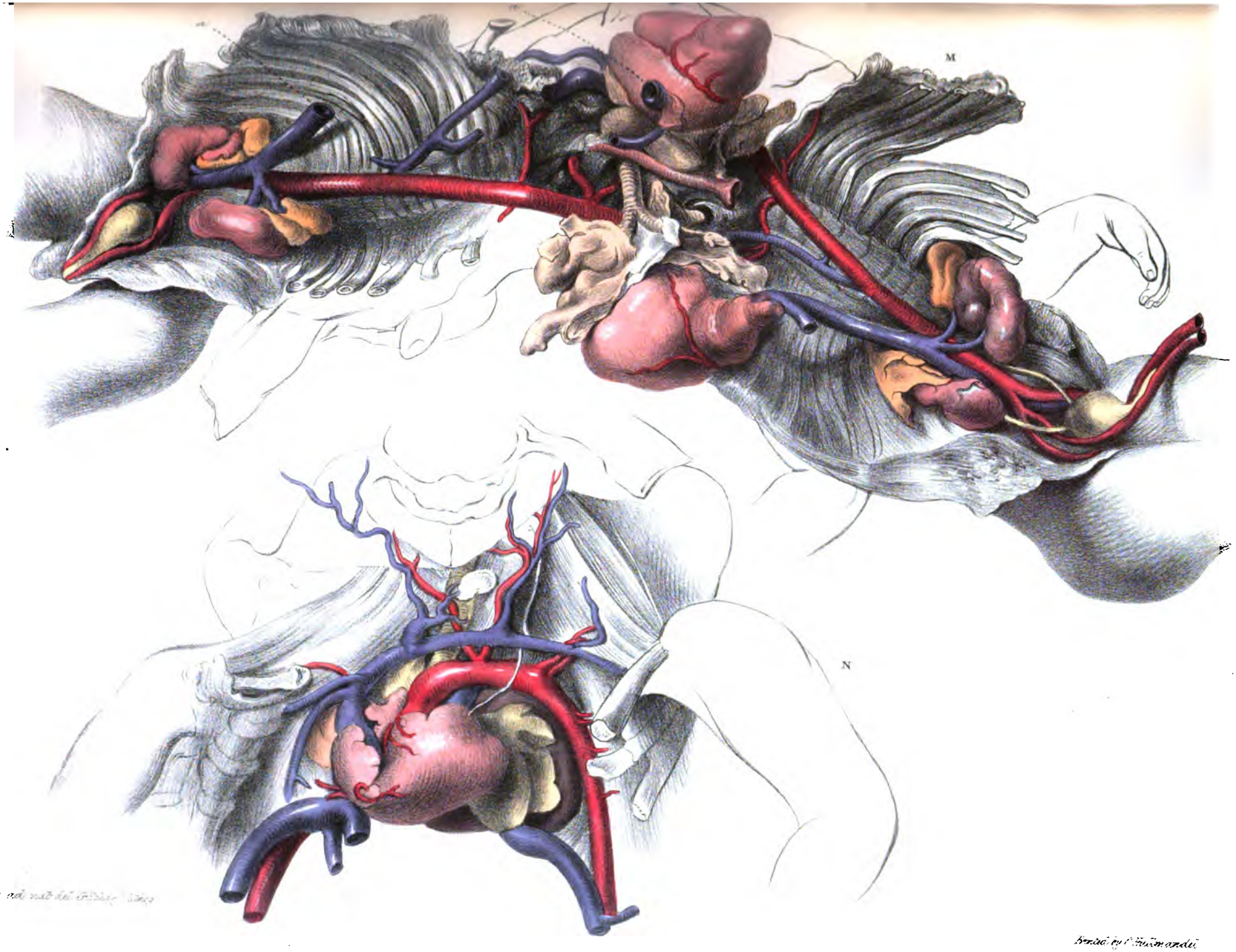

M
N

VIII. *On the Examination of a Hybrid Digitalis.*

By the Rev. J. S. HENSLOW, M.A.

PROFESSOR OF BOTANY, AND SECRETARY TO THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read Nov. 14, 1831.]

Although the propagation of hybrid plants has been much attended to of late years by several Horticulturists in England, their experiments, for the most part, seem to have been undertaken for the sole object of encreasing the forms of beautiful flowers, or of modifying the flavour of delicious fruits. But the more curious and important physiological facts elicited by the phenomenon of hybrid productions do not appear to have received a proportionate degree of attention from those who have been engaged in these experiments. Chance having favoured me with a hybrid Digitalis during the past summer (1831), in my own garden, I employed myself, whilst it continued to flower, which was from June 19 to July 22, in daily examining its characters and anatomizing its parts of fructification. I was careful to compare my observations, with as much patience and accuracy as I can command, with the structure of its two parents. It seemed to me not unlikely that something interesting might result from a rigorous examination of this kind, or at least that its recorded details might serve as a point of departure for future observations.

The plant in question was undoubtedly a seedling from a specimen of D. lutea. I have this species and D. purpurea alone of the genus cultivated in my garden, where several plants of each had been allowed to scatter their seed, and the seedlings to grow wherever they chanced to come up. I had already remarked a singularity in the general appearance of one of these, and was watching the expansion of its flowers, when I was agreeably surprized to find it to be a decided hybrid, obviously having most of its characters exactly intermediate between those of purpurea and lutea. I had no doubt whatever of its being a seedling of lutea, from the position which it occupied in the garden: in coming up amidst several plants of this species in a spot where an old plant had grown the year before; neither had any plant of purpurea grown in the same border. Besides which, my plant exactly agrees in most particulars with a hybrid procured by Koelreuter in 1768 from seeds of lutea fertilized by the pollen of purpurea*. His account is accompanied by a rude and inaccurate figure which by no means tallies with his own description of the plant. In general habit, this hybrid approaches much nearer lutea than purpurea, Plate xv. Fig. 1. It is however decidedly taller and more robust than any specimens of the former species which my garden ever produced. Koelreuter indeed asserts that the specimens raised by him were taller than either of their parents, but he assigns a lower limit to the height of purpurea than that to which many plants of this species have attained with me. Notwithstanding its more robust character and somewhat darker hue, the eye would scarcely have recognized, upon a mere casual observation and

* Acta Acad. Petropol. Anno 1777.

before its flowering, any peculiarity sufficiently striking to class it apart from some of the varieties of lutea, but a little closer inspection immediately detected certain decided points of difference. The whole plant is not so smooth as lutea, having a decided tendency to become downy, and being completely so on the under surface of the leaves, Plate xv. Fig. 2. The glabrous surface of lutea is one great characteristic of the species; though, if the D. rigida of Lindley* is to be considered as a variety of it, which he seems to think probable, even this character fails. A few hairs are always indeed distributed here and there in the ordinary state of this plant, and seem to indicate the possibility of a transition from the one condition to the other, dependant probably on certain circumstances of soil or situation. From the ordinary condition of the leaves of lutea, however, those of the hybrid differ in a marked manner. They are even nearly as woolly on the under surface as the leaves of purpurea.

Examination of the external characters of the Hybrid.

I shall first describe the external characters of its several organs, comparing them with those of the parent plants. In Plate xvi, the corresponding parts in the fructification of the parents and of their hybrid are arranged in three columns, those of the latter occupying the middle column. A single glance of the eye will thus be sufficient to shew how exactly intermediate most of its organs are both in size and form, and in some cases also in color, to those of the two parents. There are however some remarkable deviations from this condition, which will be presently noticed.

* Lindley Digitalium Monographia, fol. Lond. 1821.

Comparative view of the external characters of the three plants represented in PLATES XV. *and* XVI.

Purpurea.	*Hybrida* (*purpureo-lutea* *).	*Lutea.*
	PLATE XV.	
Biennial.	*Root.* Perennial, according to Koelreuter, and apparently so in the present instance, the plant having thrown out several offsets.	Bi-tri-ennial.
3—5 feet.	*Stem.* About $3\frac{1}{2}$ feet.	2—3 feet.
$1\frac{1}{2}$—3 feet.	*Raceme.* About $1\frac{1}{2}$ feet.	$\frac{3}{4}$—$1\frac{1}{4}$ feet.
less secund, and laxer.	secund, dense, nodding above.	denser.
woolly.	*Leaves.* Nearly smooth above, quite woolly below.	glabrous.
very soft.	Somewhat soft.	firmer.
crenato-dentate.	Dentate.	dentate.
petiolate, oblong.	*radical*, sub-petiolate, broadly-lanceolate, Fig. 2.	somewhat narrower.
	caulinar, sessile, narrower.	
broader and shorter.	*Bracteas*; Lanceolate.	narrower & longer.
longer than the Calyx and frequently than the bracteas.	*Pedicels.* About the length of the Calyx, and and somewhat shorter than the bracteas.	shorter than the Calyx and much shorter than the bracteas.
large, cernuous.	*Flowers*, medium size, nearly horizontal.	small, more drooping.
	PLATE XVI.	
I.	II.	III.
more spreading.	1. *Calyx*, moderately spreading in flower, afterwards connivent.	less spreading, at length more closed.
broader.	*a. sepals*, ovato-lanceolate, the odd one much narrower.	narrower.

* If a general rule for naming Hybrids should be thought advisable, perhaps it will be found convenient always to prefix the name of the plant which supplies the pollen to that which furnishes the ovule.

Purpurea.	*Hybrida* (*purpureo-lutea.*)	*Lutea.*
more hairy.	*b.* hairy on the margins.	less hairy.
purple.	2. *Corolla,* Yellow ground tinted with red.	yellow.
spots more numerous, deep purple, and rings paler.	A few dark purplish-red spots surrounded by a paler ring in the throat and tube.	no spots.
less hairy.	Smooth, with hairs in the mouth.	more hairy.
obscurely 4 lobed, the upper emarginate.	Distinctly 4 lobed, the lobes blunt, the uppermost notched.	4 lobes deeper, acute, the upper deeply notched.
half the length, convergent.	3. *a. Stamens* length of the tube, nearly parallel.	somewhat more extended and divergent, according to Koelreuter; but I could see no very appreciable difference.
deeper orange-yellow, with numerous spots often confluent.	*b, c: Anthers* yellow inclining to orange, with a few small scattered purple spots.	lighter yellow, no spots.
much more oblique.	Oblique to the filament, converging above.	
	d, e: Pollen White, elliptic when dry, and sperical when moist. Some of the grains obscurely three-cornered, many are abortive, but those perfected are of exactly the same size and shape as in purpurea and lutea, being somewhat less than $\frac{1}{1150}$ of an inch in diameter.	
	4. *a. Pistil,* covered below with small glandular hairs.	
few hairs.	*style* cylindrical, with a few hairs on the lower part.	hair reaches higher up.
much more acute.	*b. stigma* cloven, very obtuse.	more acute.
more ovate and more pubescent.	*c. ovarium* oblong, pubescent.	more acute and less pubescent.
much more numerous.	*d. ovules* numerous, and exactly of the same shape and size as those of purpurea and lutea.	much less numerous.

Commentary on some parts of the preceding comparison.

Raceme. Although one of the characters of lutea lies in the very decidedly secund position of the flowers, some plants have them disposed in a squarrose manner round the axis.

1. *Calyx.* About one half the number of the flowers of the hybrid had five sepals and the other half six, (Plate XVI. II. 1. *c.*) and the sections given (from *d.* to *l.*) represent the different modes of their arrangement. Figs. *d.* and *h.* however appear to be their normal condition in æstivation, the other modifications having probably resulted from inequalities introduced during the expansion of the flower. The occasional development of a sixth sepal seems to be no uncommon occurrence in this genus, and I have met with it several times in specimens of lutea and ferruginea.

2. *Corolla.* In the colored copies of Professor Lindley's monograph, there are two varieties of lutea (see his Plates XXIV and XXV) in which the corolla is tinged with red. One of these (Plate XXIV) he considers to be a hybrid plant. In shape and size it approaches very nearly to the subject of the present paper, but the other (Plate XXV) more closely resembles lutea. In his figure of lutea also, (Plate XXIII) there is a little tinge of red in the mouth of the tube, on each side the base of the lip. I have never myself found the slightest tinge of red in any specimen of lutea, though the yellow is deeper and more inclining to orange in the parts above mentioned. If however it should be quite certain that genuine specimens of lutea do occur with a tinge of red in any part of their corolla, this circumstance must considerably modify our speculations as to how far the present hybrid may have derived this color from the male parent.

Flowers of lutea are not unfrequent with the lower lip notched (Fig. γ), which indicates the presence of a supernumerary petal blended into the tube of the corolla. In about half a dozen instances I even found this petal quite free, (Fig. β) and I believe occupying the same position as the sixth sepal in the anomalous cases just referred to. In D. ferruginea, however, I have sometimes found a sixth sepal and a notched lip in the same flower. These anomalies may therefore be considered analogous phenomena among the supernumerary developments of the two organs.

3. *Pollen.* In comparing the action of the three pollens when immersed in water, I observed all the phenomena usually attendant on this experiment, to take place in those of purpurea and lutea: their grains quickly swelled and their granules were exploded in the form of a dense cloud (Fig. F and ζ). Two kinds of granules were also observed, the smallest and most numerous of which were too minute for me to be able to ascertain their precise shape and dimensions by the highest powers of my instruments; the others, much fewer in number, were considerably larger, and lay dispersed among the smaller like pellucid spots on a darker ground; and these might even be distinguished through the coats of the grains before their expulsion had taken place. Some pollen of purpurea taken from a withering stigma exhibited very distinctly the presence of the exserted membranous tubes (*boyaux*) described by A. Brogniart, Amici, and others, in the Ann. des Sciences, (Fig. G). Some of the granules also were marked on the surface by three blotches (Fig. H). Grains of pollen taken from the hybrid readily swelled upon immersion in water, though most of them appeared to be void of granules. Some few however certainly contained the larger

kind of granules, and I could see their explosion accompanied by successive and sudden contractions and dilatations of the grains themselves. But I could never detect any cloud of smaller granules similar to that which was exploded from the pollen of the parents, and which always proceeds from the grain by a continuous and slow emission, whereas the larger granules in the hybrid were discharged at intervals, and by separate efforts, and lay scattered at a distance from each other over the field of view (Fig. *f*.)

Koelreuter has given it as his decided opinion, derived from his numerous experiments, that true hybrids never reproduce their kind. Later experimenters have doubted this fact, and some seem to consider the question as quite settled to the contrary, at least with respect to the possibility of fertilizing a hybrid by the pollen of one or other of the parent species. But in prosecuting this enquiry we must be very cautious to keep in view the perfect distinctness of the two questions, whether it be *probable* and whether it be *possible* that hybrids should reproduce their kind. If it be *possible* that a true hybrid may do so, it may still be very *improbable*, from some deficiency in that connection of circumstances, of whatever description it be, which is essential to secure the fertilization of the ovule. We might imagine* for instance, so great a discrepancy to exist between the respective circumstances suited to the healthy action of its vegetative and reproductive functions, that although one climate may be adopted for securing the former, another might be required for obtaining

* This hypothesis is thrown out merely in the way of illustration, and not as likely to afford any solution of the cause of infertility observable in Hybrids, at least in most of

the latter, and thus the plant might continue to grow and flourish in one latitude, and yet be incapacitated for ripening its pollen or perfecting its ovules unless it could also thrive upon removal to another. There are certain plants, considered to be hybrids, which undoubtedly reproduce their kind freely enough; but some of these at least, if not all of them, are mere varieties of the same species. Thus Koelreuter ascertained that all the plants raised between D. purpurea and D. thapsi, by fertilizing the ovules of either by the pollen of the other, were constantly prolific, but then he also ascertained that D. thapsi itself when cultivated by him, after five generations assumed all the characters of purpurea. He consequently rightly inferred that D. thapsi was to be considered no otherwise than as a Spanish variety of the more common form of the species. If, again, it were possible for a true hybrid to be fertilized by the pollen of either of its parents, though it could produce no fertile pollen for itself, it would then evidently be in much the same condition as the female plant of any dicecious species, and its fertility might be secured by the instrumentality of insects, &c. In the present plant I repeatedly observed that the blossom always fell before the anthers on the shorter stamens had burst; and in order that this should not operate in diminishing the chance of impregnation, I touched some of the stigmas with the pollen extracted from these anthers, but without any success. Possibly however the pollen was not sufficiently ripened. I also touched other stigmas with the pollen of purpurea, and others again with that of lutea; but all these experiments failed in fertilizing any of the ovules. Koelreuter was equally unsuccessful in his attempts to fertilise this hybrid. I must here record what has appeared to me a remarkable circumstance, brought before my notice during

the prosecution of these enquiries. There were three or four plants of lutea in my garden which were quite deficient in pollen, and which nevertheless produced perfect seeds. I was unable to detect even a single grain of pollen either healthy or abortive in their anthers, though these latter organs appeared to be well formed and perfected. The ovaria of these plants indeed contained plenty of ovules, most of which I afterwards observed had been fertilized since their seeds ripened. These plants must therefore have been fertilized by the pollen of other specimens in their neighbourhood; at least according to all our present notions on this subject. But then the ovules of the hybrid were also similarly circumstanced, and if they had been capable of receiving the same influence from other plants, there is no apparent reason why they should not have proved fertile also.

4. *Ovules.* In the parent plants, the ovules begin to grow and develop themselves immediately after the fall of the corolla, whilst in the hybrid they soon wither away. It is remarkable however that all symptoms of decay in the ovarium are strictly limited to the ovules themselves, for even the little protuberances upon which they are seated on the placenta remain succulent, as do the various parts of the pericarp, including also the base of the style: all which continue healthy and attain their perfect dimensions, the valves alone slightly collapsing from the deficiency of the ovules in the enlarged cells. Plate xvii. Fig. 4. But the stigmatic tissue dries up, and a cavity is thus left through the upper part of the dissepiment, forming an opening between the two cells, Fig. 5. *e.* The same effect sooner or later takes place also in the seed vessels of the parents.

Recapitulation. In reflecting upon the points of resemblance and of disagreement in the organs of fructification of these three plants,

the most striking circumstance which we have hitherto noticed in their external characters, is the perfect identity in size and shape both of their pollen and of their ovules. As the respective organs which contain these bodies, viz. the anthers and the ovaria, are each proportionate to the different sizes of the three flowers themselves, it is evident that a flower of lutea must have much less pollen and many fewer ovules than one of purpurea, which in fact the most casual observation is sufficient to shew. The ovules of the hybrid also are about intermediate in number to those produced by the parents. It will be a subject worthy of future investigation, to determine whether one condition necessary for securing the hybridity of two species, require their pollen and ovules to be of the same, or of nearly the same dimensions. Except in the above instances, and in the very peculiar shape of the stigma, all the other external characters of the hybrid appear to be precisely intermediate between those of its parents. The chief physiological difference observable in the external economy of the organs of fructification seems to reside in the fall of the corolla, which in the parents does not take place till after the anthers have discharged their pollen and become perfectly withered, whereas in the hybrid the corolla falls before the anthers on the shorter stamens have burst, and when even those on the longer pair, although opened, have hardly parted with their pollen, and have not as yet become in the least withered. The style and stigma of all three appeared to comport themselves alike, that is to say, they all began to wither soon after the fall of the corolla.

Examination of the internal structure of the Organs of Fructification.

Before I begin the detail of this examination, I may at once state, that so far as I have hitherto been enabled to pursue it, I have not perceived the slightest difference between the internal structures of the three plants, and as their organization is somewhat different from any of the cases selected by Mons. A. Brogniart to illustrate his paper on the formation and developement of the embryo, the present attempt may not be without some general interest to the physiologist, independent of the objects connected with the particular enquiry for which it has been undertaken. The method which I pursued was always to examine the various parts dissected, first, in specimens of purpurea, and then to compare them with the like parts in hybrida, and lutea. Though it is possible therefore that I may accidentally have overlooked some defect and dissimilarity in the internal structure of the hybrid during this common and simultaneous examination of all the three, and may have represented in the drawings some appearance or other strictly belonging only to the anatomy of purpurea, yet I do not think such an error could very probably have occurred. As the main object in view was the direct comparison of the three plants, any striking difference at least would have been noticed, and the subject have been submitted to a rigorous re-examination.

Vessels of the Pistil. Plate XVII. Fig. 1. represents a longitudinal section of the ovarium perpendicular to the dissepiment, and consequently passing through both the cells; and Fig. 2. is another longitudinal section, at right angles to the last, and through the plane of the dissepiment, or rather, it represents the surface

obtained by tearing the ovarium asunder down the thickness of the dissepiment, which is composed of two skins with parenchymatous matter between them. The threads of vascular tissue arranged in a circle round the axis of the pedicel (a), after giving off veins to the calyx and corolla (b), and again to the pericarp (c), diverge on either side into the placentæ (d), a little above its lowest point, and then ramify or subdivide through its substance into separate fibres (d') which proceed directly to the bases of the ovules. Fig. 3 represents a transverse section of the upper part of the ovarium with the lower part of the style, the valve which is nearest the spectator being removed, as also are the ovules in this cell. The smaller veins (d'), of which more than twenty are seen rising through the pericarp, all terminate in the base of the style; but the two larger ones (c), which run along the dehiscidal edge of the pericarp, rise through the whole length of the style. The stigmatic tissue (e), (Fig. 1, 2, 3) descends down the middle of the style till it comes into contact with the summit of the placenta. When the appearances here represented are examined with the highest magnifiers, then more intimate structure is exposed, as in Plate xviii. where Figs. 1 and 2 are two transverse sections of the pistil, of which the former corresponds to one quarter of the circumference of the ovarium represented in the lower part of Fig. 3. Plate xvii., and the latter agrees with the section through the style in the upper part of the same figure. Plate xviii. Fig. 3. and 4. are longitudinal sections of the same organ, the former through the stigma, the latter through the summit of the ovarium where the stigmatic tissue (e) descends to the placenta, as in Fig. 1. Plate xvii. In these highly magnified sections all the corresponding parts are designated by the same letters as in the former figures.

The veins (c), &c. are also composed of tubes of trachea, which in the larger veins (d) are very numerous. I have counted sometimes between thirty and fifty of them in the construction of a single vein (c), a fact which would not be suspected upon a casual observation, but which becomes evident by digesting the style in nitric acid, when these elementary parts are easily separated. Their terminations are in the form of elongated beaks, and they all end together a short distance below the stigma. (See Plate xvii. Fig. 3.) The other elementary parts of all these veins are certain extremely delicate tubes, which invest the central bundle of trachea, and give it the appearance of being surrounded by a mucous or gluteous substance, but which under the highest powers of the microscope may be separated into these tubular vessels, whether subdivided or not by transverse diaphragms, I was unable to satisfy myself. This very delicate tissue has the same general appearance as the stigmatic tissue, which in these plants descends down the centre of the style, to the summit of the placenta. Where this latter tissue terminates in the stigma, it is indeed evidently composed of distinct cells easily separable from each other by nitric acid. Plate xviii. Fig. 3. (a) Lower down however the cells are more elongated (n), and lower still, where this tissue meets the placenta, I could neither detect any transverse diaphragms in it, nor even detach its cells (of which it was such) from each other at their extremities by the action of nitric acid, though they were easily separated longitudinally into long filamentous strings. In this part of its course therefore the stigmatic tissue appears rather to be tubular than cellular in its structure. After this tissue has become divided into two bundles, penetrating on either side through the dissepiments into the two cells, it seemed to me, upon a most careful examination, to coat

over the whole surface of the placenta. It is very difficult however to be quite certain of this fact, and I may be wrong; but after numerous dissections made upon the three plants, I found I could generally raise, with the point of a very fine needle, a thin gelatinous film of delicate fibrous structure from between the ovules Fig. 4. (s), which film seemed to be similarly constituted, and also continuous with the stigmatic tissue (e).

Cellular tissue of the Petal. The cells are for the most part compressed into tolerably regular rhomboidal dodecahedrons, excepting in the placenta, where, as the ovarium increases, the vesicles assume that irregular character as well described and represented by Mons. A. Brogniart in the parenchyma of the leaf, (Ann. des Sc. Vol. xxi.) and they have the same sort of interstices filled with air between them as those which occur in that organ. When the style is digested in nitric acid, the separate vesicles of its cellular tissue become cylindric-oval, Fig. 5. (b): and I have represented an appearance (p) which was noticed several times upon some of these vesicles, of a faintly marked band running down one side. Further examination may perhaps throw some additional light upon this circumstance, but at present I know not to what cause it may be ascribed.

Epidermis of the Floral Organs. Plate xvii. Fig. 6, 7. The flattened cells are of the same size in the three plants, their diameter being somewhat more than the thousandth of an inch. They vary in shape from hexagonal to quadrangular prisms bordered by straight, or waved sides. This membrane is irregularly supplied with stomata (r). When digested in nitric acid, the cells assume the appearance represented in Fig. 7., as though the granular matter they contain were coagulated into a nucleus, or else were enclosed in a separate internal vesicle. Whether this

appearance originate in any optical deception, I could not sufficiently satisfy myself; but if, as I am inclined to think, it does not, the fact must have been hitherto overlooked from the difficulty of detecting the true plane of junction between the contiguous cells, owing to the very great transparency of their membrane. Thus, in Fig. 6, where this epidermis is less magnified, the cells appear to be separated from each other by anastomosing veins or canals, whilst in Fig. 7. it is shewn that their true planes of junction run directly along the middle of these canals. I am however quite positive upon another point which has been a subject of dispute among physiologists; I mean the existence of a delicate homogeneous membrane investing this epidermis. Such a membrane may be distinctly separated by the action of nitric acid, from the epidermis of the corolla, filament, and style. It is faintly marked by parallel longitudinal striæ Fig. 7, (*g*), and appears to coat over the whole surface of these organs, but whether it is perforated by a fissure opposite each stoma I did ot ascertain.

Structure of the Filament. Plate XVII. Fig. 8, 9. The cellular tissue of this organ consists of elongated rhomboidal dodecahedrons, as the elongated hexagons seen in its longitudinal section sufficiently explain (Fig. 9.). A single bundle of tracheæ runs up the middle of it, invested by the peculiarly delicate fibrous tissue already noticed.

Structure of the Anthers. Plate XVII. Fig. 10—12. The fibrous cells* composing the inner coat of the anther, appeared to me quite as distinct and perfect in the hybrid as in the parents. Nor did I observe the slightest difference in the formation and

* See Purkinje "De cellulis antherarum fibrosis, &c. 4to. Vratislaviæ 1830."

condition of any part of this organ in either of the three plants. In general, a transverse section shewed the fibrous-cells to be arranged in a triple tier (Fig. 10.). These curious vessels seemed to be set, as it were, upon the sides and edges of void dodecahedral and other polyhedral spaces, as though certain original cells of these shapes had disappeared and left this framework of their structure alone standing. The triple tier is not distinguishable upon looking directly down upon the inner surface of the anther (Fig. 11.), but some of the fibrous-cells may be seen standing upon the junction-edges of the cells of the epidermis, where this membrane has been partially cleaned of the inner coating composed of them. Fig. 12. (*k*) is the appearance which they assume when detached by digestion in nitric acid: (*k*) being the cells of the epidermis, (*l*) an accidental appearance in a grain of pollen recalling somewhat of the character of the grain figured at Plate XVI. Fig. 3. H.

Structure of the Ovules. Plate XVII. Fig. 13. When the corolla is Expanded, the ovules are entirely composed of a congeries of large vesicles, and their surface has a very remarkable and granulated appearance. At this period of their existence I was unable to detect any thing very precise respecting the distinction and distribution of their several parts. The foramen (*m*) however was evidently seated near the hilum, and a darker spot indicated the chalaze (*n*) to be at the opposite extremity (see also Plate XVIII. Figs. 1. and 4.) In the ovules of purpurea and lutea, there is no difficulty in tracing the separate parts of the ordinary structure, if they be examined shortly after their impregnation, but before their fertility is secured I have not hitherto been able to detect in these plants, more than in the hybrid, any thing but a homogeneous mass of cellular tissue.

Possibly I have not given this part of the investigation sufficient attention. When the ovules are digested in nitric acid, the detached cells assume an oval shape, Fig. 15. (*o*), and are yellowish. But among them I several times observed a larger cell (*p*) which was more transparent and whiter, and which I fancied might be the origin of the embryonic sack. These component parts are best exhibited by crushing the ovule between two flat pieces of glass. Fig. 14. represents a monstrosity in which an ovule was observed to stand upon a sort of pedicel.

Recapitulation. So far then as these researches have hitherto proceeded in comparing the internal structure of the floral organs of the hybrid with those of its parents, no appreciable difference has been detected. The elementary vesicles of which their cellular tissue is constructed seem to be all of the same size, and consequently it is evident that fewer of these vesicles must be employed in the conformation of any of the parts of hybrida, and still fewer in those of lutea, than in completing the corresponding parts of purpurea. But there appears to be nothing actually defective in any part of these organs in the hybrid, nothing wanting of whatever is to be found in those of the two parents. The nutritive apparatus more especially, so far as we have examined it, seems to be quite perfect, and as the functions performed by it in all three plants are precisely the same up to the period when the flower falls, there seems to be no reason for suspecting the hybrid to differ in any particular from its parents in the perfection of its conservative organs. Since however the functions of the reproductive apparatus appear to cease in the hybrid before they do in the parents, it should seem that there must be some deficiency in this part of its organization, though it has not yet been noticed. Should the Society con-

sider the details of this examination worthy their attention, I propose to myself the further satisfaction of prosecuting it afresh next summer, if another opportunity should be permitted me. Indeed I ought to add, that in the present state of this enquiry, so little additional light has been thrown upon the great questions connected with the phenomenon of hybridity, that I should hardly have felt myself justified in presenting these remarks to their notice, were it not in the hope that they might save some time and trouble to whomsoever may be inclined to take up the subject, and possess the means of carrying on the investigation of it still further.

DESCRIPTION OF THE PLATES.

PLATE XV.

THE raceme (Fig. 1.) and radical leaf (Fig. 2.) of the Hybrid.

PLATE XVI.

The various parts of the floral organs in the three plants contrasted together. The details are at pages 4 and 5.

As the same parts in the three columns are marked by corresponding letters in three alphabets, viz. in Roman capital, small Italic, and Greek characters, it will be unnecessary to refer to more than the figures in one compartment for the purpose of explaining those in the others.

1. *Calyx.* A. sepals separated and spread open: B. their marginal hairs magnified: *c.* with supernumerary sepal: *d.* to *l.*, arrangement of the sepals during inflorescence. N. B. These sections do not refer to the arrangement of the sepals in *æstivation*, which by some neglect I omitted to notice.

2. *Corolla.*

β. with supernumerary petal: γ. ditto blended with the tube and forming a notched lower lip.

3. *Male Orgdns.* A. is of the natural size; the rest are more or less magnified.

A. Position of the stamens in the tube of the corolla: B. a front, and C. a back view of the anthers: D. dry, and E. moistened grains of pollen, lying on squares representing the $\frac{1}{1150}$ of an inch: F. a grain exploding upon the application of moisture: G. three grains taken from off the surface of a withering stigma, with their tubes (*boyaux*) exserted: H. a grain with three lighter blotches on the surface.

4. *Female Organs.* A. is of the natural size; the rest are more or less magnified.

A. pistil: B. stigma: C. transverse section of the ovarium: D. an ovule at the period of the flowers expansion, placed on a micrometer divided to the $\frac{1}{500}$ of an inch.

PLATE XVII.

Anatomy of the parts of fructification. All the figures excepting Fig. 4, are more or less magnified. The same letter is always employed to designate the same parts in the different figures.

Fig. 1, 2. *Ovarium*, longitudinally divided; in the first case perpendicular to, and in the second down the plane of the dissepiment.

a. The pedicel with its circle of vascular bundles surrounding the axis: *b*, branches of this circle given off to the calyx and corolla: *c*, two larger bundles which run up the pericarp, along the future line of its dehiscence, and rise through the whole length of the style: *d*, separation of the vascular bundles into two bands which enter the two lobes of the placenta near their base, and rising through their substance *d'*, again separate and subdivide, giving off single vessels to the bases of the ovules: *e*, the stigmatic tissue descending through the style to the summit of the placenta.

Fig. 3. A transverse section through the summit of the ovarium, and again through the base of the style. The valve and ovules of one cell are removed. The letters designate the same parts as in the last Figure, with the additions of *c'*, small vascular bundles rising through the pericarp, all of them terminating in the base of the style.

Fig. 4. Ripened pericarp of the Hybrid, of the natural size.

Fig. 5. The same magnified, with one valve removed—exhibiting the dissepiment, and one lobe of the placenta, which is still fleshy, and covered by abortive ovules: *e* a cavity left by the drying up of the stigmatic tissue.

Fig. 6. *Epidermis* of the corolla, with a glandular hair and two stomata (*f*).

Fig. 7. The same digested in nitric acid and more highly magnified; *g*, being the investing pellicle faintly but very regularly striated.

Fig. 8, 9. *Filament*; transverse and longitudinal sections.

Fig. 10. *Anther*; a section perpendicular to its coats, exhibiting the triple tier of its fibrous cells.

Fig. 11. A fragment of the coats of the anther viewed on the inside perpendicularly to its surface, which is partly divested of the fibrous-cells.

Fig. 12. Details of the anther after it has been digested in nitric acid; *h*, fibrous-cells *k*, vesicles of the epidermis; *l*, a grain of pollen peculiarly marked.

Fig. 13. *Ovule; m*, foramen; *n*, chalaze.

Fig. 14. Monstrosity of ditto.

Fig. 15. Details of the ovule after digestion in nitric acid; *o*, the smaller vesicles composing the bulk of the ovule; *p*, a paler colored vesicle occasionally found among the former.

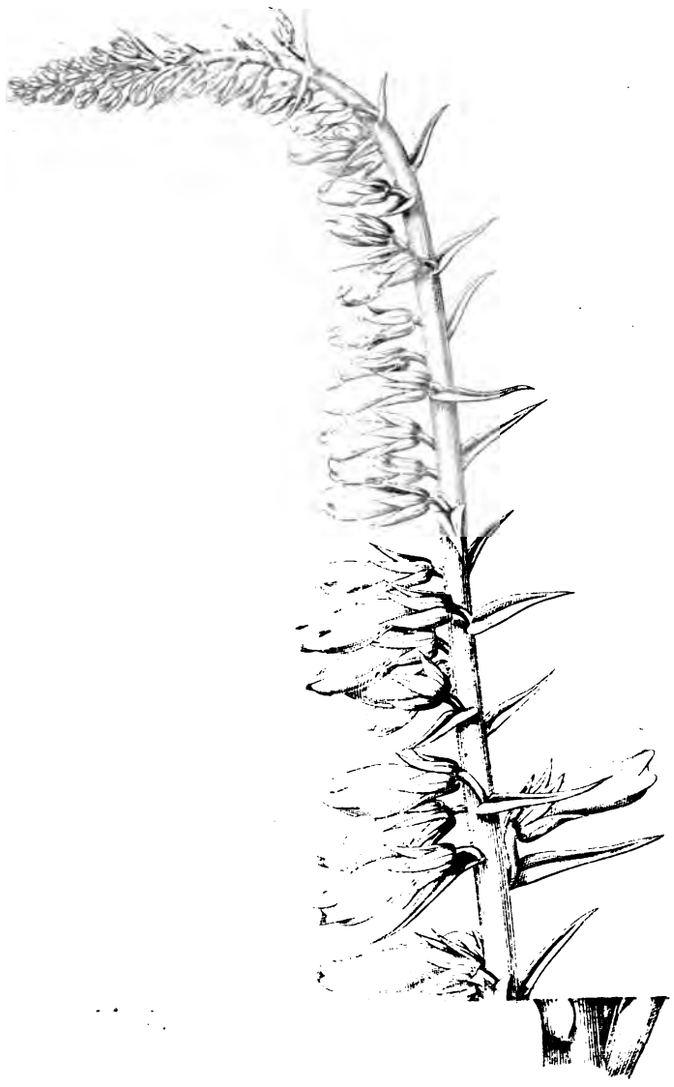
PLATE XVIII.

Highly magnified sections of the style and ovarium. Wherever the same letters are used in this plate as in the last they designate the same parts.

Fig. 1. Transverse section of one quarter of the upper part of the ovarium. Fig. 2. Transverse section of the style. Fig. 3. Longitudinal section of the stigma, and part of the style. Fig. 4. Longitudinal section of the base of the style and apex of the ovarium, perpendicular to the plane of the dissepiment.

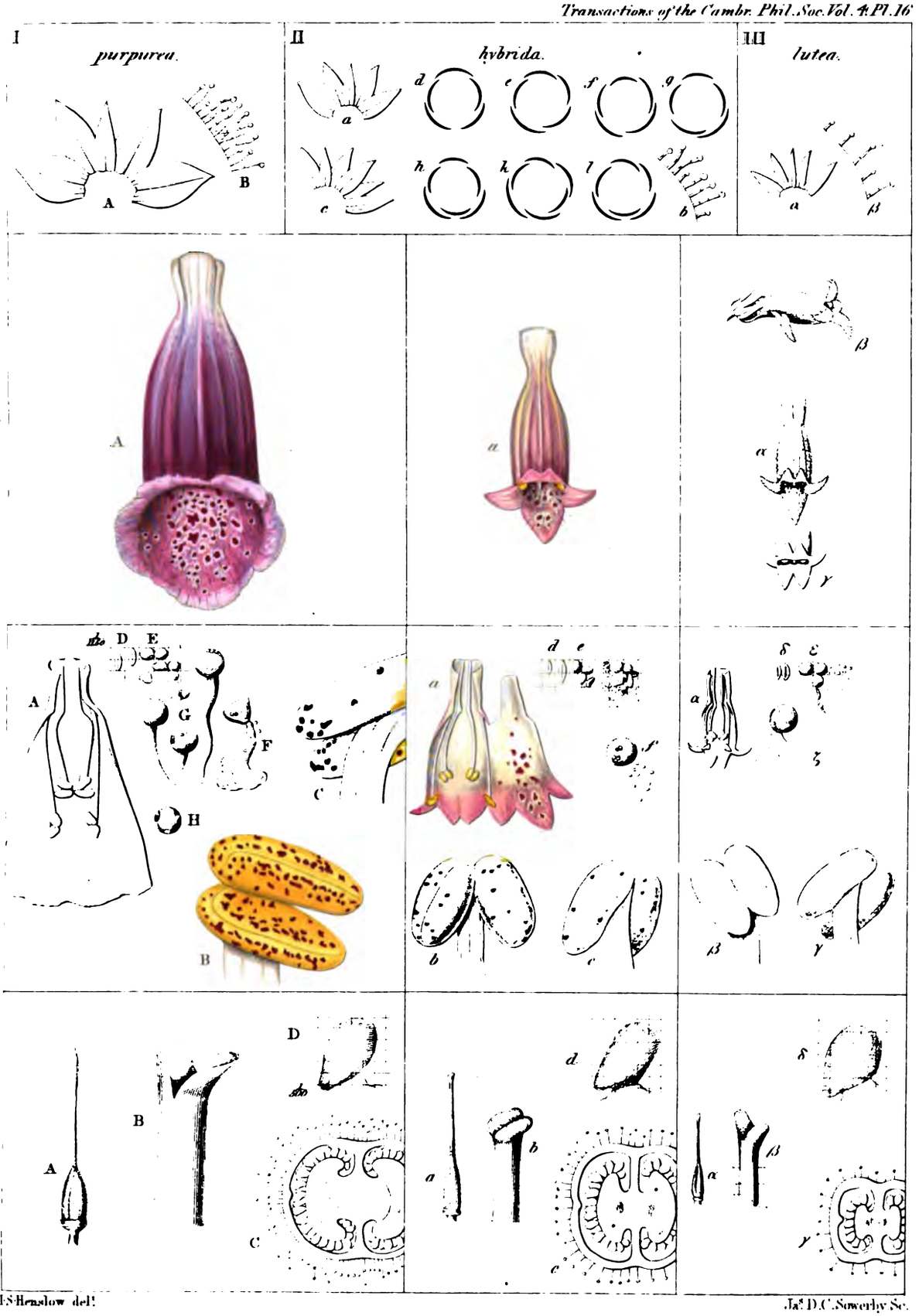
c, the two large veins, or bundles of tracheæ, which rise through the whole length of the style: *c'*, the numerous smaller veins which terminate in its base: *d'*, fragments of the vascular bundles which rise into the placenta and branch off to the ovules: *e*, stigmatic tissue descending down the centre of the style to the summit of the placenta; *e'*, the same tissue coating over the *surface* of the placenta, and passing round the bases of the ovules: *m*, foramen, and *n*, chalaze, indicated by darker spots; and in Fig. 1. the position of a raphe is apparent through the ovule, by a darker band extending from the hilum to the chalaze: *q*, vesicle of the stigma: *r*, tubular vesicles of the stigmatic tissue.

Fig. 5. *o*, Vesicles of the cellular tissue of the style detached by digestion in nitric acid: *p*, one of them marked by a transverse band, when seen more highly magnified.



Digitalis hybrida
(purpureo-lutea)

C. Henslow del.



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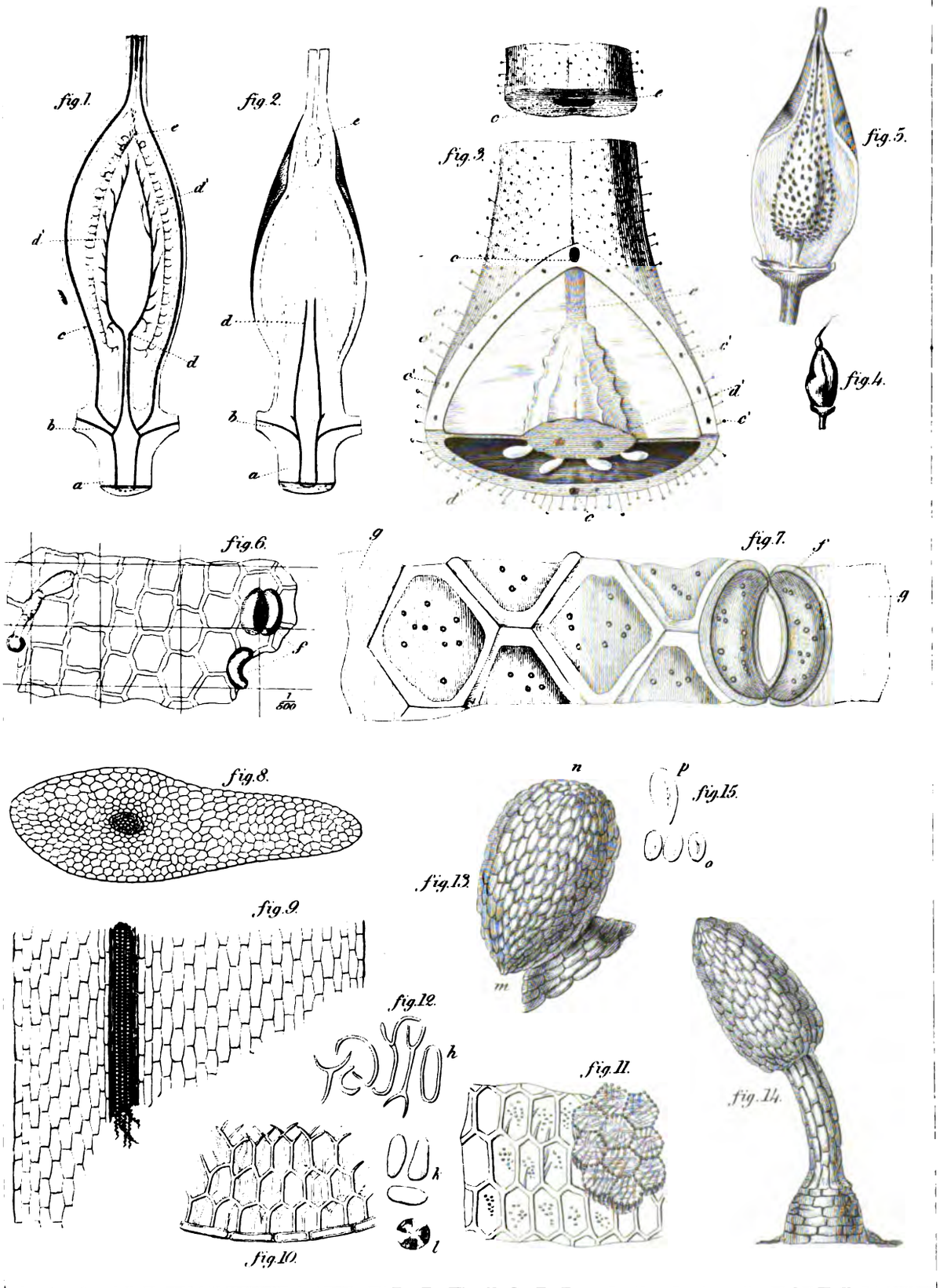
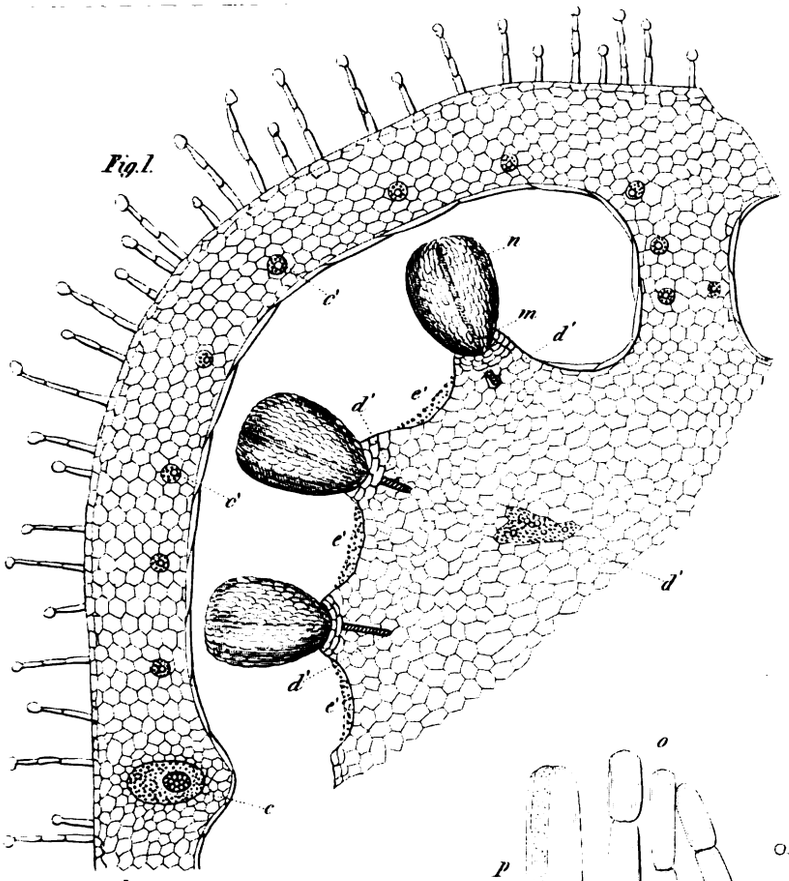


L. S. Handow, del.t

Jas. D. C. Sowerby, sculp.t



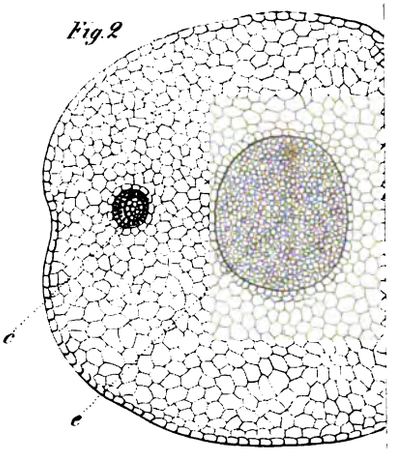


Fig. 5.

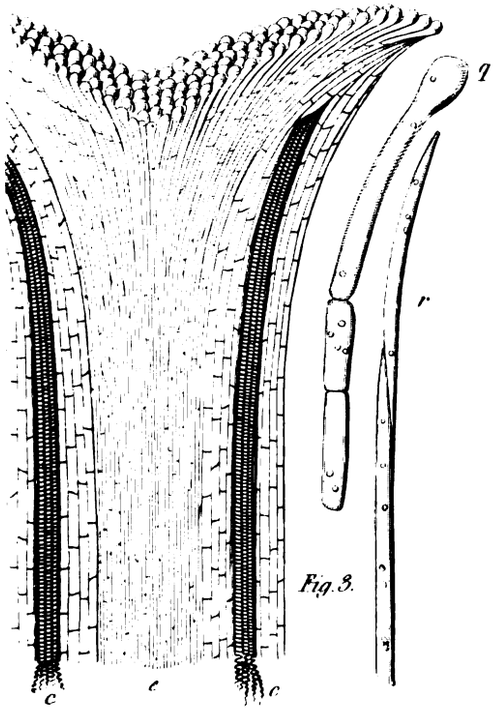


Fig. 3.

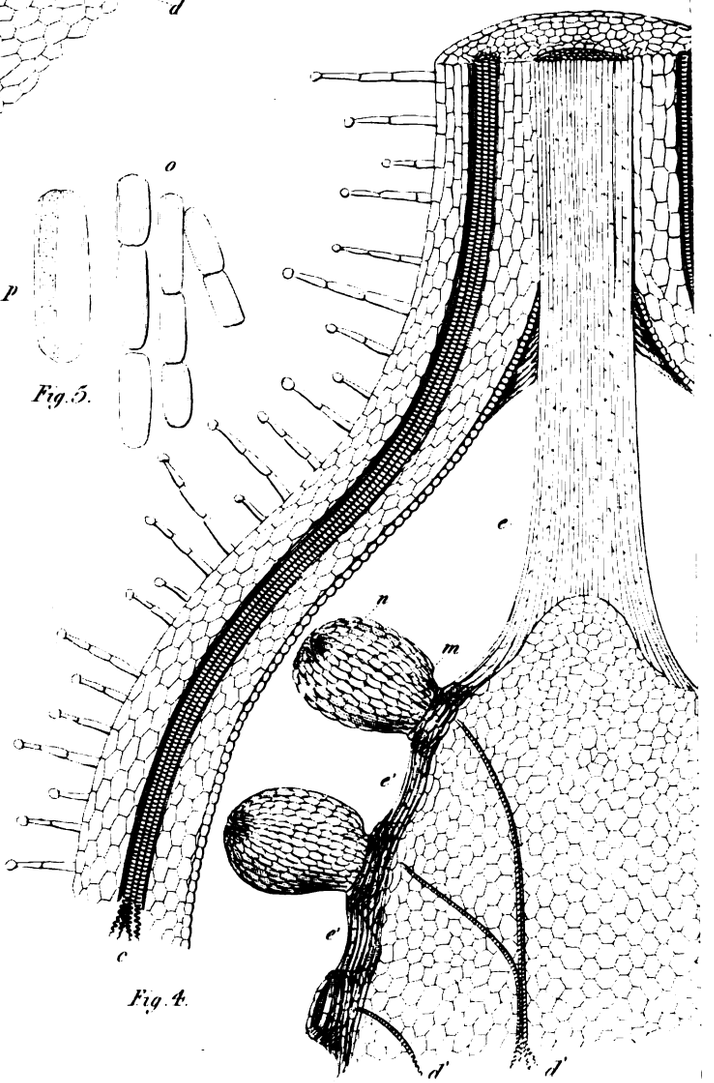


Fig. 4.

J. S. Henslow del.

Jas. D. Sowerby, sculp.

IX. *On a remarkable Modification of Newton's Rings.*

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[Read Nov. 14, 1831.]

THE following variation of the observation of Newton's coloured rings will it is hoped be considered as conclusive, so far as it goes, in favour of the undulatory theory of light. It was suggested by a consideration of the values assumed in particular cases by Fresnel's general formula for the intensity of reflected light; experiment has entirely confirmed my anticipations: and the fact appears to be perfectly inexplicable on any theory of emissions.

To begin with a case generally known, suppose that a convex lens of great focal length is placed on another convex lens, or on a plane glass, or on a concave glass where the radius of concavity is greater than the radius of the convexity which rests upon it: and suppose common light to fall on it, and to be received by the eye after reflection. A set of rings is seen with a remarkably black spot in the center: considerable pressure being sometimes necessary to insure the blackness of the central spot. On inclining the incident ray, the rings dilate, but the center remains perfectly black, and continues so till the direction of the incident light is parallel to the upper surface of the lens.

Now instead of common light, suppose that polarized light is incident. The general appearances are not altered; the only modification being that, if the plane of polarization is perpendicular to the plane of reflection, when the angle of incidence becomes equal to the polarizing angle the whole of the reflected light disappears, and with it the whole system of rings. But on increasing the angle of incidence the reflected light again appears, and the system of rings is restored, with the center black as before. It is indifferent whether the light is polarized before incidence, or a polarizing substance (as a plate of tourmaline) is placed between the lenses and the eye.

Instead of placing the lens on a plate or lens of glass, let it be placed on a polished plate of metal. If common light is used, the rings are seen as before (though not so black), but the central spot is perhaps not quite so large as before. On increasing the angle of incidence up to 90°, the rings dilate as before, but the central spot, I think, diminishes a little (at least in proportion to that formed when the lens is placed on a glass plate). In this case then, the general character of the appearances is almost exactly the same as before.

If however the lens is placed on a polished plate of metal, and if the light is polarized either before or after incidence* in

* I have carefully verified this assertion (that it is indifferent whether the light is polarized before or after reflection) because I think that it leads to important theoretical conclusions. If polarization were a *modification* of light (as Dr. Brewster and others have supposed), it might be conceived that polarization before incidence might destroy its power of producing rings at a certain angle, or might change the tints; but when the reflexion is performed, and the rings are actually visible to the eye with a dark center, it seems quite inconceivable that any *modification* or *physical change* in the light should make that center appear white. The satisfactory explanation is, that polarization is a *resolution* of the vibrations into two sets at right angles to each other, performed in such a manner that the two sets can in general be separately exhibited, and that in this instance only one is transmitted to the eye.

a plane perpendicular to the plane of incidence, the phænomena undergo a remarkable change. When the angle of incidence is small, the central spot is dark, but not black. It continues dark and without any sensible alteration of size as the angle of incidence approaches to the polarizing angle of the glass. The rings become faint (the central dark spot retaining the same magnitude as long as it is visible) and disappear* when the angle of incidence is equal to the polarizing angle. On increasing the angle of incidence by a very small quantity the rings are again seen, of the same magnitude, but the central spot without any sensible alteration of size is now *white*. And the intensity and colour of the rings appear to be, in every part, complementary to what they were before. This state continues with very little alteration in the magnitude of the central white spot, till the angle of incidence becomes 90°.

If the light is polarized before or after incidence in the plane of reflexion, there is no such change of colours. The magnitude of the central spot is altered; but through all variations of the angle of incidence the central spot still remains darker than the ring which immediately incloses it.

If common light is incident at an angle greater than the polarizing angle, and a plate of tourmaline is held between the glass and the eye, the axis of the tourmaline being in the plane of reflection, the central spot is black. On turning it to the right or the left, the dark spot dilates and a white spot arises in the center, which acquires its maximum diameter when the axis of the tourmaline is perpendicular to the plane of reflexion. On

* This simple fact (the disappearance of the rings while abundance of light is reflected from the metal) seems to be satisfactory evidence, if any were wanted, to shew that the rings are produced by interference only.

turning farther, the white spot contracts and vanishes, and the dark spot diminishes till the axis of the tourmaline is again in the plane of reflexion.

The plate of metal used in these experiments was a telescope speculum, with tolerably bright surface. To guard against all sources of error from want of contact of the lens and the metal, a broad flat ring of thick sheet lead was placed on the lens, and was sometimes loaded with a weight of about six pounds, distributed over its circumference. The changes in question therefore had no connexion with the changes of colour in the central spot when the lens and glass plate are not perfectly in contact. And it will easily be seen that they have nothing in common with the change from black to white produced in some of Sir W. Herschel's experiments (Phil. Trans. 1807 and 1809), the latter being merely a substitution, by a kind of slight of hand, of the rings of transmitted light for those of reflected light.

The explanation of this phænomenon will be found in the expression for the intensity of rings produced by the interference of two streams of reflected light, the quantity of light reflected being expressed by Fresnel's formula. If ι be the angle of incidence within the glass and ι' the corresponding angle of refraction, and if the light be polarized in the plane perpendicular to the plane of reflexion (that is, if the vibrations of the particles of ether take place wholly in the plane of reflexion), and if the extent of displacement in the vibrations before incidence be represented by $a \cdot \sin \frac{2\pi}{\lambda}(vt-x)$, then that in the vibrations after reflexion from the second surface of the glass will be

$$a \cdot \frac{\tan(\iota-\iota')}{\tan(\iota+\iota')} \cdot \sin \frac{2\pi}{\lambda}(vt-x).$$

This stream of light interferes with that reflected from the surface of the metallic reflector: in which the expression for the displacement is $a.A.\sin\frac{2\pi}{\lambda}(vt-x'-B)$. In this formula x' differs from x by the difference of paths described by the two rays, or rather by the space in air equivalent to that difference; which, if T be the thickness of the plate of air, is $2T\cos\iota'$. Thus the displacement by the metallic reflexion is $aA.\sin\frac{2\pi}{\lambda}(vt-x-2T\cos\iota'-B)$, where A and B are probably functions of ι' whose form is unknown. We can assert however that when ι' is small, A is positive and B is not great (else, as will be seen from the subsequent expressions, there would be a white spot at the center): and the gradual change of appearances, except at the polarizing angle, makes it probable that B is always small. At any rate there is not the slightest reason to believe that A or B, which depend only on the properties of the metal, would undergo any sudden change exactly at the polarizing angle of the glass.

Now the peculiar phænomenon which is the principal subject of this paper is thus explained. The central spot, where $T=0$, is produced by the composition of two displacements

$$a\frac{\tan(\iota-\iota')}{\tan(\iota+\iota')}\sin\frac{2\pi}{\lambda}(vt-x)$$

and $a.A.\sin\frac{2\pi}{\lambda}(vt-x)$ (considering $B=0$).

As ι' is greater than ι, the former of these is negative when ι is small, and the latter is positive. Consequently these displacements partly or entirely destroy each other, and the center is dark. But at the polarizing angle, $\iota+\iota'$ (by Brewster's law) $=90°$, and $\tan(\iota+\iota')$ is infinite; the first expression vanishes, and the

only light that comes to the eye is that reflected from the metal, unmixed with the other light. Beyond the polarizing angle $\iota+\iota'$ is $>90°$, $\tan(\iota+\iota')$ is negative, and $a\dfrac{\tan(\iota-\iota')}{\tan(\iota+\iota')}$ is positive: the two displacements have the same sign, and therefore are added together, and the central spot is therefore bright.

... To shew distinctly that the character of the rings as well as that of the central spot ought to change at the same time, it will be best to take the general expression. We have to add together

$$a\cdot\frac{\tan(\iota-\iota')}{\tan(\iota+\iota')}\sin\frac{2\pi}{\lambda}(vt-x)$$

and $a\cdot A\cdot\sin\dfrac{2\pi}{\lambda}(vt-x-2T\cos\iota'-B)$.

The sum is

$$a\left\{\frac{\tan(\iota-\iota')}{\tan(\iota+\iota')}+A\cos\frac{2\pi}{\lambda}\cdot\overline{2T\cos\iota'+B}\right\}\sin\frac{2\pi}{\lambda}(vt-x)$$

$$-aA\cdot\sin\frac{2\pi}{\lambda}\cdot\overline{2T\cos\iota'+B}\cdot\cos\frac{2\pi}{\lambda}(vt-x).$$

If this be put in the form

$$P\sin\frac{2\pi}{\lambda}(vt-x-Q),$$

it is easily seen that

$$P^2=a^2\left\{\frac{\tan(\iota-\iota')}{\tan(\iota+\iota')}+A\cos\frac{2\pi}{\lambda}\cdot\overline{2T\cos\iota'+B}\right\}^2+a^2A^2\cdot\sin^2\frac{2\pi}{\lambda}\cdot\overline{2T\cos\iota'+B}$$

$$=a^2\left\{\frac{\tan^2(\iota-\iota')}{\tan^2(\iota+\iota')}+A^2+2A\cdot\cos\frac{2\pi}{\lambda}\cdot\overline{2T\cos\iota'+B}\cdot\frac{\tan(\iota-\iota')}{\tan(\iota+\iota')}\right\}.$$

Now P^2 represents the intensity of the light in the mixture: consequently the last formula will represent that intensity, or the intensity of light that comes to the eye. It is easily seen

that if ι and ι' are small, ι being $< \iota'$, this is *minimum* when $2T\cos\iota' + B = 0$, or $= \lambda$, or $= 2\lambda$, &c. and *maximum* when

$$2T\cos\iota' + B = \frac{\lambda}{2}, \text{ or } = \frac{3\lambda}{2}, \text{ \&c.:}$$

and that this continues till $\iota + \iota' = 90°$: but as soon as $\iota + \iota'$ exceeds 90°, the expression is *maximum* when $2T\cos\iota' + B = 0$, or $= \lambda$, or $= 2\lambda$, &c., and *minimum* when

$$2T\cos\iota' + B = \frac{\lambda}{2}, \text{ or } = \frac{3\lambda}{2}, \text{ \&c.}$$

If B be small, the first of these represents rings similar to Newton's reflected rings, and the second represents rings similar to his transmitted rings.

If the light is polarized in the plane of reflection, the expression for the displacement in light reflected from the lower surface of the glass is $a.\frac{\sin(\iota-\iota')}{\sin(\iota+\iota')}$, which does not change sign like the former: and a similar train of reasoning shews that the character of the rings which it produces will not suddenly change.

If the light is polarized in a plane inclined by the angle α to the plane of reflection, we must regard this as consisting of two vibrations, one perpendicular and one parallel to that plane, whose proportions are $\cos\alpha : \sin\alpha$. The original vibration perpendicular to that plane must now be taken

$$a.\cos\alpha.\sin\frac{2\pi}{\lambda}(vt-x),$$

and that parallel to the plane

$$a.\sin\alpha.\sin\frac{2\pi}{\lambda}(vt-x).$$

The intensities of the reflected streams are found to be:

For the light polarized in the plane of reflexion,

$$a^2 \cos^2 \alpha \left\{ \frac{\sin^2 (\iota - \iota')}{\sin^2 (\iota + \iota')} + A'^2 + 2A' \cdot \cos \frac{2\pi}{\lambda} \cdot \overline{2T \cos \iota' + B} \cdot \frac{\sin (\iota - \iota')}{\sin (\iota + \iota')} \right\}$$

(where A' and B' are quantities analogous to A and B, and where it is probable that A' is always positive and B' not very great).

For the light polarized perpendicular to the plane of reflexion,

$$a^2 \sin^2 \alpha \left\{ \frac{\tan^2 (\iota - \iota')}{\tan^2 (\iota + \iota')} + A^2 + 2A \cdot \cos \frac{2\pi}{\lambda} \cdot \overline{2T \cos \iota' + B} \cdot \frac{\tan (\iota - \iota')}{\tan (\iota + \iota')} \right\}.$$

The intensity of the compound light which comes to the eye is the sum of these. The value of T corresponding to the middle of a bright or dark ring is determined by making the sum a maximum or minimum with respect to the variation of T. This gives,

$$\tan \frac{2\pi}{\lambda} \left(2T \cos \iota' + \frac{B + B'}{2} \right)$$

$$= \frac{A' \cos^2 \alpha \cdot \cos (\iota - \iota') - A \sin^2 \alpha \cdot \cos (\iota + \iota')}{A' \cos^2 \alpha \cdot \cos (\iota - \iota') + A \sin^2 \alpha \cdot \cos (\iota + \iota')} \tan \frac{\pi}{\lambda} (B - B').$$

If $\tan \frac{\pi}{\lambda} (B - B')$ is positive, and $\cos (\iota - \iota')$ negative; upon increasing α from 0 to 90° the value of $\tan \frac{2\pi}{\lambda} \left(2T \cos \iota' + \frac{B + B'}{2} \right)$ increases to infinity and then becomes negative and decreases: that is $\frac{2\pi}{\lambda} \left(2T \cos \iota' + \frac{B + B'}{2} \right)$ increases: and therefore the diameter of the ring increases. Upon increasing α from 90° to 180° the opposite change takes place. This represents exactly the fact of

observation; and it proves therefore that B is $> B'$*; a conclusion of some importance, as shewing that plane-polarized light reflected at a metallic surface becomes elliptically polarized, and as connecting these phænomena with those of a very different kind discovered by Dr. Brewster. It appears here that the phases of vibrations in the plane of reflection are more retarded, than those perpendicular to that plane.

If $\cos(\iota+\iota')$ is positive, that is, if the angle of incidence is less than the polarizing angle, it appears that upon increasing α the diameters of the rings ought to decrease. But it is easily seen that the change ought to be much less than in the former case. For in the former case, if one value of $\frac{2\pi}{\lambda}\left(2T\cos\iota'+\frac{B+B'}{2}\right)$ is $n\pi+\beta$, when $\alpha=0$, then, upon increasing α to 90^0, $\frac{2\pi}{\lambda}\left(2T\cos\iota'+\frac{B+B'}{2}\right)$ changes through $n\pi+90^0$ to $n\pi+180^0-\beta$, and the whole change is therefore $180^0-2\beta$. But in the latter case $\frac{2\pi}{\lambda}\left(2T\cos\iota'+\frac{B+B'}{2}\right)$ changes from $n\pi+\beta$ through $n\pi$ to $n\pi-\beta$, and the whole change is therefore 2β, which is exactly supplemental to the former whole change. Now in Newton's rings formed between two lenses, β is 0; from the general similarity of the rings formed by a metal reflector when $\alpha=0$, it is certain that β is small; consequently though the dilatation of the rings in the former case depending on the increase $180^0-2\beta$ in $\frac{2\pi}{\lambda}\left(2T\cos\iota'+\frac{B+B'}{2}\right)$ is considerable, the contraction in the latter case (depending on the decrease 2β)

* If B were $< B'$, the rings would contract instead of expanding; and if B were $= B'$, the diameters of the rings would not alter, but their intensity would diminish to 0, when $\tan^2\alpha=-\frac{A'}{A}\cdot\frac{\cos(\iota-\iota')}{\cos(\iota+\iota')}$, and rings of the opposite character would then appear.

is small. I have submitted this to experiment, but I have not been able to discover with certainty any alteration in the size of the rings.

The reasoning upon which the principal experiment described in this paper was anticipated, may probably be applied in many similar cases. The following is a very remarkable instance. If a lens of a low-refracting substance be placed on a plate of a highly-refracting substance, or vice versâ, and if they be illuminated with light polarized perpendicular to the plane of incidence, I expect that while the angle of incidence is less than the polarizing angle of the low-refracting substance, the central spot will be dark; when the angle of incidence is greater than the polarizing angle of this substance and less than that of the other, the central spot will be bright; and when the angle of incidence is greater than the polarizing angle of the highly-refracting substance, the central spot will again be dark. I have not yet procured any substances proper for the verification of this conclusion.

G. B. AIRY.

OBSERVATORY,
June 21, 1831.

X. *A Monograph on the British species of Cyclas and Pisidium.*

By the Rev. LEONARD JENYNS, M.A. F.L.S.

and Fellow of the Cambridge Philosophical Society.

[Read Nov. 28, 1831.]

The genus *Cyclas* of Bruguiere was instituted for the reception of certain species of "Bivalve Mollusca" inhabiting fresh water, which were associated by the older Linnæan authors, either with *Cardium* or *Tellina*. Only three of these shells appear to have been distinctly known to Montagu as natives of this country, who referred them in his "Testacea Britannica" to the former of the two genera just mentioned. Other indigenous species have been discovered in later years, some of which have been described, and from time to time been made known to the public; nevertheless, from want of having their characters accurately defined, and still more from not being illustrated by figures sufficiently large to convey a correct idea of their relative proportions, it is not always very easy to identify them, nor to determine how far such species are really distinct from one another, or from those before known. It may be added also, that the synonyms have been much confused, and the same name has been applied in several instances to more than one species. It is on these grounds that I have been induced to draw up the following

paper. Having for many years paid considerable attention to this family, during which time I have not only increased the list of British species, but have also endeavoured to ascertain the extent of variation to which each of them is subject; having likewise been fortunate in obtaining authentic specimens of many of those alluded to by British authors, I thought that it might be rendering a service to the conchologists of this country, if I were to throw together, in the form of a monograph, the observations which I had made, and to draw up an amended list of all the species hitherto detected in this island, accompanied by full descriptions, and illustrated by magnified figures. In my attempt to do this, I have not merely considered the general characters of the shell, but in determining the species have derived much assistance from attending to the animal inhabitant. The species themselves are found in rivers, ditches, and streams; the smaller ones not unfrequently in the gullies that are cut in pastures for the purpose of draining the soil: they all however live readily in confinement for several days when kept in water; and under these circumstances their different habits may be conveniently observed. Occasionally it will be found that they become languid and inert, especially if they have been confined a long time; but they may generally be roused into activity by the sudden application of cold spring water, and this is by far the best method of getting a sight of the siphonal tubes, which in some cases afford good distinguishing characters. Indeed it is absolutely requisite to caution conchologists against drawing any conclusions with respect to the specific distinction of these animals from a mere inspection of the shell alone. This is so liable to vary from age, peculiarity of situation, and probably from other causes, that it becomes necessary in some cases to compare a

large number of specimens, collected from different sources, in order to determine the characters of a single species with any degree of precision. Occasionally the shell becomes exceedingly ventricose at the expense of its height, which is thereby considerably diminished; and the valves which perhaps naturally meet at an acute angle, under such circumstances meet at an obtuse one. This is particularly the case with one or two varieties hereafter to be described. Neither can sculpture be relied upon, the striæ varying exceedingly in number and distinctness according to the nature of the water in which the shell is found: a circumstance of which Dr Leach was not sufficiently aware when he formed three species out of *Pisidium amnicum*. Age likewise produces great changes: not only are young shells much more compressed than adult ones, but in many instances the relative proportions of their parts are different. Indeed in the case of the minute species, so great and general a similarity prevails amongst their young, that it is hardly possible to identify them in this state without the closest examination.

After what has been stated, I need hardly add, how cautious I have been in characterizing the minuter species of this group, and that it is not till after repeated observations upon very extensive series of each, including many varieties from different localities, that I venture to bring forward the following list of such as are found in this country, as one which I trust will be found more complete than any which has appeared hitherto.

It will be observed, that with respect to the arrangement of these shells, I have deviated from that of Lamarck and most authors in referring them to two genera. This I have done in conformity with the views of Pfeiffer, who in his excellent work

on the land and fresh water Mollusca of Germany*, first instituted the genus *Pisidium* for the reception of those species which are characterized by an inæquilateral shell, and a single siphon at the posterior extremity of the cloak. This distinction, founded upon the structure of the animal as well as that of the shell, appears to be perfectly natural; while there is little doubt that its utility will be soon acknowledged, when increased attention shall have been paid to this family, and future discovery have still further augmented the number of species, which we have every reason to expect will be the case. The propriety, however, of such a division of the old genus *Cyclas* will be best seen from the following comparative view of the characters of those adopted in this paper.

GEN. I. CYCLAS, *Lam. Pfeiff.*

Animal: Pallium anticè, pro pede exserendo, apertum, posticè connatum, et in tubum siphonalem longum, duplicem, contractilem, extensum. *Pes* linguiformis, valde extensilis.

Testa corticata, suborbicularis, subæquilateralis. *Dentes cardinales* minuti; in dextrâ valvulâ unicus plus minusve complex; in sinistrâ duo obliquè collocati. *Dentes laterales* longitudinales, compressi, lamelliformes, in valvulâ dextrâ subduplicati. *Ligamentum* externum, posticum, lateri longiori insitum.

GEN. II. PISIDIUM, *Pfeiff.*

Animal: Pallium anticè, pro pede exserendo, apertum, posticè connatum, et in tubum siphonalem brevem, simplicem, contractilem, extensum. *Pes* linguiformis, valde extensilis.

* *Systematische Anordnung und Beschreibung deutscher Land- und Wasser-Schnecken, &c. von Carl Pfeiffer. Cassel, 1821.*

Testa corticata, subovalis, inæquilateralis. *Dentes cardinales* minuti, in dextrâ valvulâ unicus plus minusve complex; in sinistrâ plerumque duo. *Dentes laterales* longitudinales, compressi, lamelliformes, in valvulâ dextrâ duplicati. *Ligamentum* externum, posticum, lateri brevi insitum.

Although Pfeiffer has the merit of having first separated the above genera, his characters are not clearly defined, from the circumstance of his having confounded the anterior and posterior parts of the shell. The extremity from which the siphonal tube is protruded, and which strictly speaking is the posterior, he has termed differently in his descriptions of the two genera. I have adopted the views of Blainville with reference to this point, as well as in the selection of terms employed in characterizing the species, and accordingly it is to his "Manuel de Malacologie" that I must refer for an explanation of all such as occur in the following paper. The best characters for distinguishing the above genera are drawn from the structure of the siphonal tube, and the position of the hinge considered relatively to the two extremities of the shell. In the genus *Cyclas*, the tube is capable of protrusion to a considerable length, and although single at the base, is always divided at the apex, the upper portion, which is generally the shorter of the two, acting as the anus, while the lower, which is both longer and has a larger aperture, serves to conduct the water to the branchiæ. In *Pisidium*, this tube is single and undivided throughout its whole length, and although to a certain degree admitting of extension and contraction, is never protruded to the distance that it is in *Cyclas*. Indeed in both genera it appears to be quite a matter of pleasure whether it be exserted at all; the animal being often seen, both at rest and in motion,

with the tube either entirely concealed within the shell, or extended to its utmost limit. Another obvious distinction between these two genera, is afforded by the position of the hinge and cardinal teeth, with respect to the extremities of the shell. The teeth themselves are similar in the two instances, at least in general character; although subject to minute variation of form, more especially the cardinal tooth in the right valve, even amongst individuals of the same species; but the situation of the hinge is essentially different in *Cyclas* and *Pisidium*. In the former genus it is nearly central, the *posterior* portion of the shell being only to a slight degree longer than the anterior: in the latter, it is placed towards one end, and the *anterior* portion is obviously the longest; and although the excess of length in the first section of this genus is not very considerable, yet it will always be found in front of the hinge and not behind it. This will be made to appear more clearly by inspecting Plate XIX. in which Fig. 3. represents the hinge of *Cyclas calyculata*, and Fig. 4. that of *Pisidium amnicum*. In each case the right and left valves are distinguished by the letters *A* and *B* respectively, whilst *a* points out the relative position of the cardinal teeth.

Having made the above observations upon the generic distinctions afforded by these shells, I shall now proceed to characterize the species themselves in the order of their respective affinities.

GEN. I. CYCLAS.

Sp. 1. C. *rivicola*, Leach.

C. testâ globoso-ovali, ventricosâ, striatâ; umbonibus obtusis; anticè lunulâ flavâ impressâ; ligamento cardinali conpicuo.

Long. 10½ lin. Alt. 8½ lin. Crass. 6½ lin.

Cyclas rivicola, *Leach MSS.—Lam. An. sans Vertèb.* 5. 558.—*Pfeiff. Land-und Wasserschn.* 121. *t.* 5. *f.* 3–5.—*Turt. Conchyl. Brit.* 248. *t.* 11. *f.* 13.—*Turt. Man. of Brit. Land and Freshw. Shells.* 12. *f.* 1.—*Flem. Brit. An.* 452.

Tellina cornea β, *Mat. and Rack. Linn. Trans.* 8. 59.—*Turt. Conch. Dict.* 180.

Cyclas cornea, *Draparn. Hist. des Moll.* 128. *t.* 10. *f.* 1—3.—*Brard, Hist. des Coq.* 219. *t.* 8. *f.* 2. 3.

Jun. C. æquata, *Shepp. MSS. Brit. Mus.*

Animal mihi ignotum.

Testa globosa, subovalis, ventricosa, solidiuscula, eleganter et distinctè striata; fusco-virescens, fasciis 2—3 saturatioribus, margine basali luteo; intus cærulescens: umbones tumidi, pallidiores, lineâ nigricante plerumque circumscripti: margo dorsalis anticè lunulâ, posticè fissurâ distinctè impressâ, utrâque flavescenti: ligamentum cardinale conspicuum.

This species is at once distinguished from the next, and from all the other British ones of this family by the superior size of the shell: the animal I have not seen. It appears to be confined to rivers, and is found I believe abundantly in the Thames, as well as in some other parts of the country. The largest specimens in my possession are from the Trent in Nottinghamshire. In its young state, it appears to be identical with the *C. æquata* of the Rev. R. Sheppard, according to specimens so named by that gentleman in the British Museum.

Sp. 2. C. *cornea*, Lamark.

C. testâ suborbiculari, globosâ, tenerrimè striatâ; umbonibus obtusis; ligamento cardinali inconpicuo.

Long. 6½ lin. Alt. 5 lin. Crass. 4 lin.

Cyclas cornea, *Lamarck*, 5. 558.—*Pfeiffer*, 120. *t.* 5. *f.* 1, 2.—*Nilsson*, *Hist. Mollusc. Suec.* 96.—*Turt. Conch. Brit.* 248. *t.* 11. *f.* 14.—*Turt. Man.* 13. *f.* 2.—*Fleming*, 452.

Tellina cornea, *Linn. Syst. Nat.* 1. 1120.—*Gmel.* 3241.—*Linn. Trans.* 8. 59.—*Dillw. Cat. of Shells*, 1. 104.—*Don. Brit. Shells*, *t.* 96.

Tellina rivalis, *Mull. Verm. Hist.* 2. 202.

Cardium corneum, *Mont. Test. Brit.* 86.

Cyclas rivalis, *Draparn.* 129. *t.* 10. *f.* 4, 5.—*Brard*, 222. *t.* 8. *f.* 4, 5.

Var. β.

Testâ subglobosâ, versus marginem basalem complanatâ; umbonibus tumidis, pellucidis, valdè prominentibus.

Long. 5½ lin. alt. 4¾. Crass. 3½ lin.

Cyclas stagnicola, *Leach MSS. Brit. Mus.*

Tellina stagnicola, *Sheppard*, *Linn. Trans.* 14. 150.

Animal album, viviparum: tubi siphonales subelongati, carneo pallidè colorati; superiore subconico, aperturâ parvâ apice perforato; inferiore cylindraceo, truncato, aperturâ ampliori: pes testam longitudine paulò superans.

Testa globosa, suborbicularis, ventricosa, tenuis, levissimè striata; nunc virescenti-fusca, zonis 1—3 lutescentibus, quarum 1 plerumque marginalis latior; nunc omninò fuscescens aut lutescens; umbones obtusi; in var. β valde prominuli, quasi inflati, pellucidi: lunula vix ulla: margo dorsalis posticè liturâ nigricanti duplici sæpe notatus: ligamentum cardinale inconspicuum.

This very common species is a general inhabitant of rivers, ponds, and ditches throughout the country, and appears to thrive equally well both in running and in stagnant water. In confinement it will oc-

easionally ascend the sides of the vessel in which it is kept, and during locomotion I have observed that the tubes are either partially exserted, or entirely concealed.

The variety β agrees with the series of specimens in the British Museum named by Dr Leach *C. stagnicola*. I am inclined to believe also that it is the same with the *Tellina stagnicola* of Mr Sheppard, as the remarks made by this latter gentleman with respect to the peculiar appearance of the *umbones*, apply very exactly; and it is particularly stated that he first received specimens from Dr Leach under the above name. Nevertheless I feel satisfied that it is a mere variety of *C. cornea*, as the animal is exactly the same in the two instances; and with regard to the peculiarity of the shell, many intermediate specimens may occasionally be met with. It is necessary, however, to mention, that the name of *stagnicola* appears to have been applied by Dr Leach at different times to two distinct species. The series of shells at present so named in the British Museum, are certainly the variety of *C. cornea* now under consideration; but I possess two specimens of a shell which came originally from Dr Leach, and which have the name of *stagnicola* under them in that gentleman's own handwriting, evidently belonging to the *C. calyculata* of Draparnaud, (Var. γ. of this paper,) and I am inclined to think that it was this latter shell which was formerly sent by the Doctor to Lamarck under the above name, and considered by that author to be a mere variety, not of *C. cornea*, but of the species last mentioned.

Other varieties of *C. cornea*, besides those above-mentioned, are not uncommon. Occasionally the shell exhibits gibbosities, and the margin becomes very obtuse: at other times the valves are much compressed, and their margins meet at an acute angle. In the fens of Cambridgeshire, a small variety is not unfrequent in the turf pits almost globular, and somewhat similar both in size and shape to a pea.

I may here observe that this, and all the other species of this family breed readily in confinement, during the spring and summer months. They are probably ovoviviparous; and the young appear to remain for a certain period within the folds of the branchiæ previous to their exclusion, since many may be found of different sizes within the parent at one and the same time. They have the faculty of producing long before they are arrived at their full growth, and even some individuals which are themselves so immature as to possess hardly any of the distinguishing characters of the species, frequently contain young of a sufficient size to be seen from without through the transparent valves.

In distinguishing this and the last species, authors have frequently drawn their essential characters from the presence and number of the longitudinal, or as they term them *transverse* grooves, indicative of the different stages of growth. But as these are very uncertain marks, depending upon age and other circumstances, I have not thought it necessary to notice them at all. The colour is not less variable.

Sp. 3. C. *calyculata*, Draparnaud.

C. testâ subrhombeâ, compressâ, tenui, albo-lutescenti, diaphanâ; natibus prominentibus, acutiusculis, tuberculosis.

Long. 5½ lin. Alt. 4½ lin. Crass. vix 3 lin.

Cyclas calyculata, *Draparn.* 130. *t.* 10. *f.* 13, 14.—*Lamarck*, 5. 559. *Pfeiffer*, 122. *t.* 5. *f.* 17, 18.—*Nilsson*, 99.—*Turt. Man.* 14. *f.* 3.

Cardium lacustre, *Montagu*, 89.

Tellina lacustris, *Linn. Trans.* 8. 60.—*Turt. Conch. Dict.* 180.

Cyclas lacustris, *Turt. Conchyl. Brit.* 249. *t.* 11. *f.* 18.

Var. β.—Tab. xix. Fig. 1.

Testâ orbiculato-rhombeâ, minus compressâ, subdiaphanâ, fusco-rufescente.

Long. 4½ lin. Alt. 4 lin. Crass. 2½ lin.

Cyclas lacustris, *Alder in Trans. Nat. Hist. Soc. Newcast.* 1. 40. *Brit. Mus. MSS.*

Var. γ.

Testâ orbiculato-rhombeâ, minus compressâ, subdiaphanâ, rufescente; natibus nigricantibus, minùs prominulis.

Cyclas stagnicola, *Leach* (*olim.*)

C. calyculata, (2), *Lamarck*, 5. 559.

Animal (in Var. β) album, tubis siphonalibus concoloribus; hi valde elongati, nunc superiore, nunc inferiore alium longitudine superante, quoad formam ferè ut in specie precedenti.

Testa quam maximè variabilis, rhombea, orbiculato-rhombea, sub-ovalis, vel exactè orbicularis; plus minusve compressa, tenuis, diaphana, levissimè striata; plerumque cærulescenti-alba, zonâ marginali lutescente;—interdum fusco-rufescens, minus diaphana, apice nigricanti: nates acutiusculæ, tuberculosæ, in α et β prominentes, interdum etiam subinflexæ: ligamentum inconspicuum.

I feel satisfied that the above described shells are only varieties of one species, and all referable to the *C. calyculata* of Draparnaud. One of them is the *C. lacustris* of the British Museum, and also of Mr Alder, as I have been enabled to ascertain from specimens kindly forwarded to me by that gentleman. Var. γ, which only differs from the last in having the tubercles on the beaks not quite so prominent and well defined, I believe to be the variety, as I have already stated, originally sent to Lamarck by Dr Leach under the name of *C. stagnicola*. Mr Alder was of opinion that his shell was the *C. lacustris* of Draparnaud, but as Lamarck has referred the variety last mentioned (which differs so little from it) to the present species, and as he was acquainted with both the *C. calyculata* and *lacustris*, there can be little doubt that this last is distinct from either of the above.—Indeed I have never seen any British shell exactly answering to the *C. lacustris* of the continental

authors. Draparnaud appears to be the first who made a distinction between this shell and the *C. calyculata*, and this distinction has been since acknowledged not only by Lamarck, but also by Pfeiffer and Nilsson; but all the British specimens that have fallen under my observation with the name of *C. lacustris* attached to them, are in my opinion nothing more than mere varieties of the species under consideration.

As Muller has described only one of these two species, I consider it doubtful to which his description applies. I have therefore made no reference to his work in the present instance.

C. calyculata is much less abundant in this country than *C. cornea*. Montagu met with it in Devonshire and Wiltshire. Mr Alder finds it near Newcastle, but says that it is rare; and it has occurred sparingly to myself in two or three parts of Cambridgeshire.—Var. β, I observed last summer (1831) in considerable abundance in one pond on Bookham-Common in Surrey, and some which I kept by me alive for a few days, showed more activity than the last species, readily and frequently ascending the sides of the vessel, and walking, like *Physa Hypnorum*, on the under side of the surface of the water*. Occasionally they remained in a quiescent state at the bottom with their posterior extremity elevated, and the siphonal tubes exserted to a considerable length, often nearly equalling that of the shell itself.—Var. γ. I have received from the North of England.

In young specimens the tubercle at the apex of each valve, so characteristic of this species, is *relatively* much larger than in the adult shell.

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* This phraseology is not strictly correct, but is perhaps sufficiently intelligible. The action intended, consists in the animal extending its foot along the surface of the water with its shell immersed, and in an inverted position. In this manner, it contrives to traverse the vessel from side to side as though it were crawling along a solid plane.

Gen. II. PISIDIUM.

* *Testâ parùm inæquilaterali.*

Sp. 1. P. *obtusale*, Pfeiffer.—Tab. xx. Fig. 1—3.

P. testâ globosâ, obliquè subovali, tenuissimè striatâ; umbonibus prominulis, obtusissimis.

Long. 1¾ lin. Alt. 1½ lin. Crass. vix. 1½ lin.

Pisidium obtusale, *Pfeiffer*, 125. *t.* 5. *f.* 21, 22.—*Brown in Edinb. Journ. of Nat. and Geog. Scien.* 1. 413.

An Cyclas obtusalis? *Lamarck*, 5. 559.

Pera gibba, *Leach*, *MSS. Brit. Mus.*

Var. β.

Testâ ovato-trigonâ, ventricosissimâ, margine obtusissimo.

Cyclas obtusalis, *Nilsson*, 101.

Animal album; tubo siphonali abbreviato, subconico; pede valdè extensili, testâ dimidio et ultrà longiori.

Testa globoso-ovalis, ventricosissima, crassitudine ferè altitudinem æquanti, nitida, subtiliter striata; plerumque virescenti-nigra vel ochraceo-nigricans, zonâ marginali (junioribus latissimâ) lutescenti, interdum subaurantiâ; rariùs omnino lutescens: umbones tumidi, obtusè rotundati, paulò prominentes.

Var. β. gaudet testâ ventricosiori, margine basali obtusissimo, quò minuatur altitudo, et forma magis trigona vel ovato-trigona provenit. Hæc varietas plerumque nigricans, ochraceo plus minusve fucata.

This species, which is distinguished from all its congeners by the extreme convexity of the shell, is certainly the *P. obtusale* of Pfeiffer, and probably the *Cyclas obtusalis* of Lamarck, but from the short de-

scription given by this last author, a little doubt attaches itself to the latter synonym.—The variety β, also accords exactly with the *Cyclas obtusalis* of Nilsson. Dr Leach called it *Pera gibba*, but I do not feel certain that all the specimens on the board so named in the British Museum are referable to this species.—It occurs not unfrequently in Cambridgeshire, inhabiting small splashy pools and other stagnant waters, and I have observed that it is often to be found in company with the *Physa Hypnorum* Drap. It is by far the most active and lively species that I am acquainted with, being always in motion, and residing less at the bottom than the rest of this family. It transports itself rapidly along the under side of the surface of the water, and appears to delight much in floating masses of confervæ and other weeds. Dr Leach's specimens came, I believe, from the neighbourhood of Battersea Fields; and I have myself also met with it in other parts of Surrey.

Obs.—The measurements of this species are usually much less than those above given.

Sp. 2. P. *pusillum*, Nobis.—Tab. xx. Fig. 4—6.

P. testâ orbiculato-ovali, compressiusculâ, subtilissimè striatâ, vix inæquilaterali; umbonibus parùm prominulis.

Long. 1¾ lin. Alt. 1½ lin. Crass. 1 lin.

Tellina pusilla, *Turt. Conch. Dict.* 167.

Cyclas pusilla, *Turt. Conchyl. Brit.* 251. *t.* 11. *f.* 16, 17.—*Turt. Man.* 16. *f.* 7.

—— fontinalis, *Nilsson*, 101.—*Draparn.* 130. *t.* 10. *f.* 8—11?

—— gibba, *Alder in Trans. Nat. Hist. Soc. Newcast.* 1. 41.

Euglesa Henslowiana, *Leach MSS. Brit. Mus.*

Var. β.

Umbonibus magis prominentibus.

Var. γ.

Striis profundiùs incisis.

Animal album; tubo siphonali brevi, nunc cylindraceo, nunc subconico, margine integerrimo; pede testam longitudine paulò superante.

Testa variabilis, plerumque orbiculato-ovalis, interdum suboblonga margine dorsali recto, vix inæquilateralis; præcedenti multò magis compressa, marginibus acutis; sæpius extraneâ rubigine obtecta, quâ remotâ, apparent striæ subtilissimæ, non nisi oculo armato conspiciendæ; in var. γ. nitida, striis distinctis, profundiùs incisis: umbones subdepressi, parùm prominuli, interdum subacuti.

This species appears to be the *Cyclas pusilla* of Turton, whose description and figure in his "British Bivalves," apply with tolerable exactness. Specimens also of that shell which I received some time since from the Rev. R. T. Lowe, with the assurance that they were authentic specimens originally from Dr Turton himself, agree with mine in every essential particular, although more compressed, and with the *umbones* not quite so obtuse and prominent. Nevertheless, I am inclined to think that this name has been occasionally applied to more than one species, particularly to some of the varieties of *P. pulchellum* hereafter to be described, which I have received from one or two collectors as the shell above-mentioned. I likewise consider this species as synonymous with the *Cyclas fontinalis* of Nilsson, although I entertain some doubts as to its identity with the *C. fontinalis* of other continental authors. Draparnaud especially has comprised under this name two varieties differing so materially in size, as to render it hardly probable that they belong to the same species.

Pisidium pusillum is distinguished from *P. obtusale* by the shell being much more compressed than in that species, and by the margins of the valves meeting at an acuter angle: the hinge is also nearly central,

the *anterior* extremity still being in a slight degree longer than the posterior. It is by no means of uncommon occurrence, residing chiefly at the bottoms of drains and ditches, where I have often found it buried at a considerable depth in the soft mud. It appears to be somewhat amphibious in its habits. Nilsson observes that it is often to be met with between the bark and the wood of decayed timber in wet places, and I have myself noticed that in confinement it will frequently leave the bottom of the vessel, and ascending the sides, take up its residence immediately above the edge of the water with its shell wholly exposed. It is a tranquil species, seldom moving much about, and never walking on the under side of the surface of the water. Where it occurs at all, it is generally in profusion.

Dr Leach appears to have raised this species to the rank of a distinct genus, under the name of *Euglesa*, but it hardly shows sufficient peculiarities to warrant this step. The shell is certainly somewhat intermediate in form between that of *Cyclas* and *Pisidium*.

Sp. 3. P. *nitidum*, Nobis.—Tab. xx. Fig. 7, 8.

P. testâ orbiculato-ovali, nitidissimâ, tenuiter striatâ; umbonibus obtusiusculis, striis paucis profundioribus.

Long. 1½ lin. Alt. vix 1½ lin. Crass. 1 lin.

Animal album; siphone brevi, infundibuliformi, aperturâ patulâ, plus minùsve margine crenato, plicatulo.

Testa minimè variabilis, orbiculato-ovalis, parùm inæquilateralis; præcedenti paulò convexior, et pro ratione longitudinis altior; albo-lutescens, nitidissima, rarò aut nunquam sorde aut rubigine obtecta, tenuiter striata, striis hic illic, præcipuè 3—5 umbones transeuntibus, distinctiùs incisis; umbones obtusiusculi, dorsalem marginem paulò superantes.

I can no where find any allusion to this species, which though similar to the two last in the general form of the shell, may at once be distinguished from both, if attention be paid to the animal. I have examined upwards of an hundred specimens from different localities, and in every instance it has preserved its characters. Its chief peculiarity consists in the formation of the siphonal tube, which is regularly funnel-shaped, with the aperture very patulous, somewhat plaited at the margin, and more or less crenate.—These appearances are not always obvious, unless the siphon is protruded by the animal to its utmost extent: the mouth of the tube, which is rendered very dilatable in consequence of the plaits, then becomes fully expanded, and the irregularity of its partially reflexed margin is rendered distinctly visible.—The shell also, which is subject to scarcely any variation, is remarkable for its extremely glossy hue and cleanly appearance, rarely presenting any of that foulness with which the last and following species are so often incrusted, although found inhabiting the same ditches; from which circumstance it would seem to follow, that in the case of those species, this is due to something more than a mere extraneous deposit from the surrounding soil. It may also be distinguished by a few peculiar striæ drawn with great regularity across the *umbones* near the apex of each valve, and cut rather more deeply than the rest. This character, however, will not be seen without a close examination. It is most visible when the animal is alive and the glossiness of the shell remains unimpaired; but even then it is sometimes necessary that this should be held to the light and turned in different directions, in order that the eye may catch the appearance in question. It is, however, more or less obvious in every specimen that I have seen.

This species is widely dispersed throughout Cambridgeshire, inhabiting various situations, though seemingly partial to clear water. It is however seldom found in any great plenty.—I have also met

with it in the ditches about Battersea Fields, and in other parts of Surrey.

* * *Testâ distinctè inæquilaterali.*

Sp. 4. P. *pulchellum*, Nobis.—TAB. XXI. Fig. 1—5.

P. testâ obliquè ovali, ventricosâ, profundiùs striatâ; umbonibus obtusiusculis, simplicibus.—Fig. 1.

Long. vix 2 lin. Alt. $1\frac{1}{2}$ lin. Crass. $1\frac{1}{4}$ lin.

Var. β.—Fig. 2, 3.

Plerumque minor, testâ tenuiùs striatâ; umbonibus subacutis.

Long. $1\frac{3}{4}$ lin. Alt. $1\frac{1}{4}$ lin. Crass. 1 lin.

Pera pulchella, *Leach MSS. in Brit. Mus.*

Cyclas fontinalis, *Brown in Edinb. Journ. of Nat. and Geog. Scien.* 1. 11. *Pl.* 1. *f.* 5—7.—*Alder in Trans. Nat. Hist. Soc. Newcast.* 1. 41.

Var. γ.

Testâ obliquè ovali, tenuiter striatâ, compressâ, marginibus acutis.

Long. $1\frac{1}{2}$ lin. Alt. $1\frac{1}{4}$ lin. Crass. $\frac{3}{4}$ lin.

Var. δ.—Fig. 4, 5.

Testâ suboblongâ, ventricosissimâ, profundiùs striatâ; margine obtusissimo.

Long. $1\frac{1}{2}$ lin. Alt. $1\frac{1}{4}$ lin. Crass. $1\frac{1}{4}$ lin.

Animal album, siphone polymorpho; cylindraceo, conico apice truncato, vel obconico; nunc abbreviato, nunc in tubum gracilem subelongatum (præcipuè in Var. δ.) extenso; margine hic illic inciso, vel integerrimo.

Testa quam maximè variabilis, in α, β, et γ, obliquè ovalis, distinctè inæquilateralis, nunc ventricosior, nunc compressiuscula, plus minusve profundè striata, nitida, cinereo-lutescens, interdum autem sorde ferru-

gineâ omninò incrustata: umbones simplices projecturâ nullâ, plerumque obtusiusculi.

Var. δ. suboblonga margine dorsali subrecto, minus inæquilateralis, ventricosissima, margine basali obtusissimo.

This species was originally discovered by Professor Henslow, and sent by him many years since to Dr Leach, who gave it the above name. I have since met with it in plenty throughout Cambridgeshire, and likewise in other parts of the county. In fact it is one of the most common species*, inhabiting rivers, ditches, and likewise the smallest streams: it is rather active in its habits, frequently ascending the sides of the vessel in which it is confined, but I never observed it to walk along the under side of the surface of the water. The siphonal tube assumes a variety of appearances even in the same individual, and it is very interesting to watch, under a low power of the microscope, the striking and rapid changes of form through which it passes in a short time. It is altogether a variable species, and the shell is of a very different character in different situations; yet from the circumstance of my possessing many intermediate specimens, I feel confident that the above are only varieties.—Var. δ. is from a pond on Bookham Common in Surrey; the others are all of frequent occurrence, and are often so much incrusted over with a kind of ferruginous earth as to be entirely concealed, and to present more the appearance of seeds or small lumps of dirt than that of shells. Whether this is the effect of soil and water, as seems to me more likely, or has any thing to do with the habits of the species, is not very obvious.

* The discovery of this and some other minute species, which though of frequent occurrence remained for a long time unnoticed by Conchologists, may be attributed to the use of a peculiar net invented by Professor Henslow about the year 1815. This instrument being constructed of the finest wire gauze enables the collector to strain the water more thoroughly than by any other method previously attempted: and thereby to separate the very smallest shells from the mud in which they are immersed.

There can scarcely be a doubt that this is the *minute shell* found by Montagu (*Test. Brit.* 88.) which he confounded with the young of *P. amnicum*. Mr Alder sent me a small variety of it from Newcastle as his *Cyclas fontinalis*, and I do not feel certain that it is not the *Pisidium fontinale* of Pfeiffer whose characters in some respects accord better with this species than with *P. pusillum* already described.

My largest specimens of this shell are from the neighbourhood of Battersea Fields.

Sp. 5. P. *Henslowianum*, Nobis.—TAB. XXI. Fig. 6, 7.

P. testâ obliquè ovali, ventricosâ, tenuiter striatâ; umbonibus subacutis, projecturâ lamelliformi adornatis.

Long. 2½ lin. Alt. 2 lin. Crass. 1¾ lin.

Pera Henslowiana, *Leach* (*olim.*)

Tellina Henslowiana, *Shepp. in Linn. Trans.* 14. 150.

Pera appendiculata, *Leach MSS. in Brit. Mus.*

Cyclas appendiculata, *Turt. Man.* 15. *f.* 6.

Animal album, tubo siphonali brevi, quoad formam paulò variabili; plerumque subconico, apice truncato.

Testa obliquè ovalis, ventricosa, anticè planiuscula, distinctè inæquilateralis, tenuitèr striata, nitidè lutescenti-alba, vel cornea, sæpiùs partim præcipuè ad apicem, sorde ferrugineâ obtecta: umbones acutiusculi, projecturâ parvâ lamelliformi adornatis.

Obs. In pullis projectura medio valvularum insidet; hinc gradatim assurgit, acerescente testâ.—(Vide Fig. 8, 9.)

The discovery of this very peculiar and well marked species is likewise due to Professor Henslow, who first found it in ditches communicating with the river Cam in the immediate neighbourhood of, and also a few miles below Cambridge. Dr Leach named it after him;

but subsequently changed the name to that of *Pera appendiculata*, reserving the above specific name for another and larger shell. This last having however since proved to be a mere variety of the following, I have restored the name of *Henslowianum* to the present species, which has indeed already been described under that title by the Rev. R. Sheppard in the Linnæan Transactions.—This shell is so strikingly distinguished by the curious eave-like projection upon the *umbones*, that it cannot be confounded with any other. In quite young specimens this projection is, as it were, a small wing arising from the middle of the valves, but as the growth of the shell proceeds, this last receiving its increase principally at the basal margin, it appears to mount higher up, until at length in adult individuals it occupies quite the summit of the shell, where it appears like a small ridge or lamina rising up vertically on either side of the hinge.—In other respects this species is very similar to the last; nevertheless it is always larger.—I have met with it in two or three parts of Cambridgeshire, but it does not appear to be of very general occurrence. Sheppard found it in Suffolk.

Sp. 6. P. *amnicum*, Nobis.—Tab. xix. Fig. 2.

P. testâ ovali, ventricosâ, profundiùs sulcato-striatâ; umbonibus obtusiusculis.

Long. $5\frac{1}{4}$ lin. Alt. $3\frac{3}{4}$ lin. Crass. $2\frac{3}{4}$ lin.

Tellina amnica, *Muller*, 2. 205.—*Gmelin*, 3242.—*Linn. Trans.* 8. 60. *Dillwyn*, 1. 105.—*Turt. Conch. Dict.* 168.

——— rivalis, *Maton, in Linn. Trans.* 3. 44. *t.* 13. *f.* 37, 38.—*Donovan*, *t.* 64. *f.* 2.

Cardium amnicum, *Montagu*, 86.

Cyclas palustris, *Draparnaud*, 131. *t.* 10. *f.* 15, 16.

Cyclas obliqua, *Lamarck*, 5. 559.—*Nilsson*, 99.

Pisidium obliquum, *Pfeiffer*, 124. *t.* 5. *f.* 19, 20.

Cyclas amnica, *Turt. Conchyl. Brit.* 250. *t.* 11. *f.* 15.—*Fleming*, 458. *Turt. Man.* 15. *f.* 5.

Var. β.

Sulcis profundiùs exaratis.

Pera fluviatilis, *Leach, MSS. in Brit. Mus.*

Var. γ.

Striis leviùs impressis.

Pera Henslowiana, *Leach, MSS. in Brit. Mus.*

Animal album, siphone paulò variabili; nunc abbreviato, subconico, apice obliquè truncato; nunc elongato, cylindraceo, apice plus minùsve recurvo.

Testa paulò variabilis, ovalis, vel obliquè trigona, distinctè inæquilateralis, ventricosa, anticè planiuscula, pulchrè striata hic et illic sulcis profundioribus; cinerascenti-fusca, maculis et zonâ marginali latâ pallidioribus, interdùm nitidè lutescentibus; rariùs omnino fuscescens aut lutescens; intus cærulescens: umbones obtusi, sorde ferrugineâ ut in præcedentibus sæpe incrustati.

This species, which was first published as British by Dr Maton l. c. is at once distinguished from all the others in this genus by its very superior size. It is not uncommon in rivers and gently running streams, residing wholly at the bottom, and being partially buried in the mud, but I have not often observed it in perfectly stagnant waters. Vars. β and γ were sent to Dr Leach by Professor Henslow from the neighbourhood of Cambridge. The former of these gentlemen considered them as distinct species, and they accordingly stand in the collection at the British Museum, under the above names; but I am perfectly satisfied that they are mere varieties, differing in nothing

but the depth and number of the longitudinal furrows and striæ, than which no characters can be more variable. The shell of this species, like that of the others belonging to this section, is frequently incrusted over with a kind of ferruginous earth, which prevails chiefly at the posterior extremity. Perhaps in the present instance this circumstance is connected with the habits of the animal; which usually having the anterior half of the shell fixed in the mud, the posterior and exposed portion receives all those finer particles of the soil which drifting downwards with the stream, are thereby deposited on its surface.—The young of this species are readily distinguished from the two last, by their more compressed shell, with the umbones scarcely at all prominent and the striæ more distinct.

The foregoing list includes all the British species belonging to the above genera which I have been able to identify satisfactorily. I possess one or two other shells which appear different from all hitherto described, but not having seen a sufficient number of specimens to judge of their true characters, I should not feel authorized in admitting them as really distinct.—I mention the circumstance, however, for the sake of exciting further enquiry upon the subject.

LEONARD JENYNS.

Swaffham Bulbeck,
Nov. 14, 1831.

EXPLANATION OF THE PLATES.

PLATE XIX.

FIG. 1. *Cyclas calyculata;* Showing the general appearance of the shell and animal inhabitant in that genus.

FIG. 2. *Pisidium amnicum.* Ditto.

FIG. 3. Hinge of *Cyclas calyculata.* A, the right, and B, the left valve; *a*, the cardinal teeth.

FIG. 4. Hinge of *Pisidium amnicum.* A, B, and *a*, as before.

PLATE XX.

FIG. 1. *Pisidium obtusale.*

FIG. 2. *Ditto;* as it appears when walking on the under side of the surface of the water.

FIG. 3. *Ditto;* viewed from one end.

FIG. 4 and 5. *Pisidium pusillum.* Two extreme varieties.

FIG. 6. *Ditto;* viewed from one end.

FIG. 7. *Pisidium nitidum.*

FIG. 8. *Ditto;* viewed from one end.

PLATE XXI.

FIG. 1. *Pisidium pulchellum: a, b,* and *c,* different appearances of the siphonal tube.

FIG. 2. *Ditto,* Var. β: *d, e, f,* different appearances of the siphonal tube.

FIG. 3. *Ditto;* viewed from one end.

FIG. 4. *Ditto,* Var. δ.

FIG. 5. *Ditto;* viewed from one end.

FIG. 6. *Pisidium Henslowianum.*

FIG. 7. *Ditto;* viewed from one end.

FIG. 8. *Ditto,* young.

FIG. 9. *Young;* viewed from one end.

OBS. All the above figures are highly magnified.

Fig. 1.

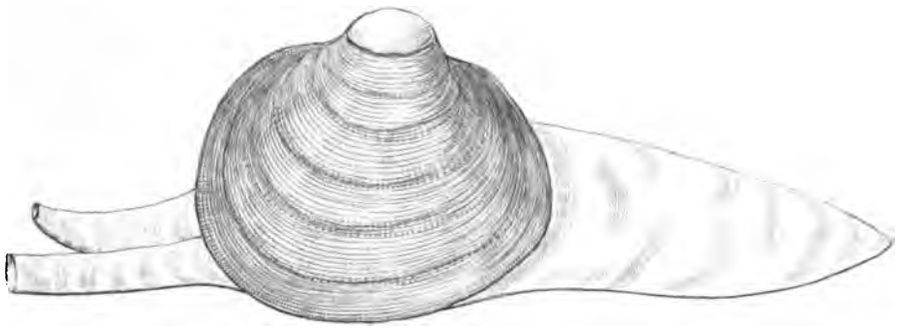

Fig. 2.

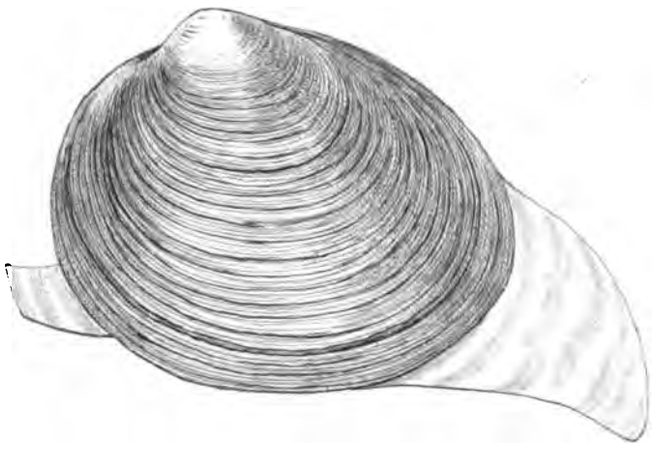

Fig.1.

Fig.2.

Fig.3.

Fig.4.

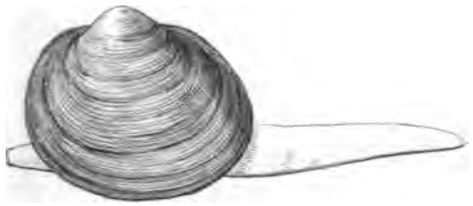

Fig.5.

Fig.6.

Fig.7.

Fig.8.

Fig. 1.

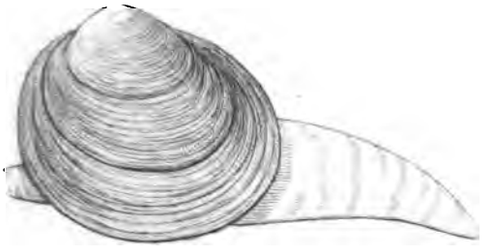

Fig. 2.

Fig. 3.

Fig. 4.

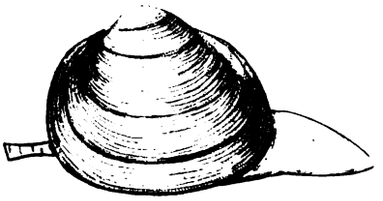

Fig. 5.

Fig. 6.

Fig. 9.

Fig. 8.

Fig. 7.

XI. *On a new Analyzer, and its Use in Experiments of Polarization.*

By G. B. AIRY, M. A. F. R. Ast. Soc. F. G. S.

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[Read *March 5*, 1832.]

On two occasions I have had the good fortune to lay before this Society anticipations of optical phænomena founded on theoretical considerations, which have been fully verified by experiment. This agreement I consider important, not because the phænomena possess any intrinsic value, but because the precise coincidence of observed appearances with theoretical calculations affords the strongest possible proof of the correctness of the theory. I have now to offer another instance of the same kind: in which the experiment was suggested solely by theoretical considerations, and in which the appearances, so far as I can observe, agree perfectly with those which theory had indicated. Like the others, it appears to me to give strong evidence of the correctness of all the fundamental assumptions of Fresnel's theory.

The experiment was first suggested by considerations of the most general kind respecting the use of the analyzing plate in the common polarizing apparatus. When polarized light (whether plane, circular, or elliptical) has passed through a crystalline plate, its intensity, by theory as well as by experiment, is the

same as before it entered. The use of the analyzing plate is to resolve the emergent light according to some general law into two parts, of which one is totally suppressed and the other (at least a definite part of it) is wholly reflected to the eye. As the crystalline plate has at incidence resolved the incident light into two parts which at emergence it has united with different degrees of retardation according to the direction in which the light was incident, the nature of the emergent light is different according to the direction in which it was incident, or (which is the same) according to the direction in which it emerges: and when resolved according to the general law above mentioned, the proportion of the reflected part varies according to that direction: and hence the various intensities of the light in different parts of the image seen with the ordinary apparatus. And this may be considered as the general explanation of the use of an analyzer, including in this term the common analyzing plate as well as such combinations as will be hereafter described.

By the common analyzing plate (an unsilvered glass reflector at the polarizing angle, or a plate of tourmaline, or a doubly refracting prism considered with reference to one pencil only) the emergent light is resolved (according to Fresnel's theory) into two sets of vibrations, one parallel to and the other perpendicular to the plane of polarization of the analyzing plate; the former of these is wholly suppressed, and the latter is wholly transmitted to the eye. And this is the only kind of analyzation, so far as I know, that experimenters or theorists have ever yet considered. But there is not the least need for confining ourselves to this method of analyzation. There are other methods, quite as simple in theory, and nearly as easy in practice, which effect a resolution of a very different kind. As the first among these I may mention

that it is easy to conceive and to treat theoretically a resolution of the light emerging from any point of the crystal into two pencils, one circularly-polarized and right-handed, and the other also circularly-polarized but left-handed. To exhibit practically the effects of this it is only necessary to contrive an analyzer which shall wholly suppress right-handed circular light and wholly transmit left-handed circular light, or vice versâ. Another more general resolution is into two elliptically-polarized pencils, one right-handed and the other left-handed, the axes of the ellipses having the same proportion, but the direction of the major axis of one coinciding with that of the minor axis of the other. I am not prepared to say whether or no any other resolution will be found practicable.

Now conceive the analyzer to be of the first new kind that I have mentioned, namely to have the power of wholly suppressing right-handed circular light and of wholly transmitting left-handed circular light, or vice versâ. I shall not consider the case of plane-polarized light incident on the crystalline plate, because the appearances are almost exactly the same (in theory as well as in experiment) as when circularly polarized light is incident on the crystal and the emergent light is analyzed by the common analyzing plate: which case I have fully considered in former memoirs in the Transactions of this Society, as well as in another work. But let us consider the case of circularly polarized light incident on the crystal, and after emergence analyzed by our new analyzer. The first idea that strikes us in this combination is, that there is nothing, except in the crystal, which has any respect to sides. For the only incident light is circularly polarized: the only light allowed to emerge is circularly polarized. The appearance therefore of the coloured rings, &c. must be such as

conveys no trace of any plane of polarization: and must not vary as the crystal or the analyzer is turned round.

In the common exhibition of the coloured rings (the incident light being plane-polarized, the analyzer being the common analyzing plate, and the inclination of the planes of polarization being any whatever) the principal trace of the planes of polarization is in the uncoloured brushes. In uniaxal crystals they form an eight-rayed star, composed of two square crosses inclined at an angle equal to that between the planes of polarization, every ray of which separates complementary rings. In biaxal crystals they compose two pairs of rectangular hyperbolas, the angle between whose asymptotes is the same as that between the planes of polarization, and whose branches divide complementary rings. The two crosses or the two sets of hyperbolas unite when the planes of polarization are parallel or perpendicular.

The first conclusion then is that, in the case under consideration, the rings exhibited by crystals will not be traversed by any brushes. Plates of Iceland spar, if they exhibit any variations of light at all, will exhibit circular rings without a cross: and plates of biaxal crystals will exhibit complete lemniscates without any interruption from curved brushes.

The next conclusion is that the brightness of the light at the poles of the image will depend only upon this consideration; whether the direction of the circularly polarized light which is incident on the crystal is the same as the direction of that which the analyzer can transmit to the eye, or is the contrary. If it is the same, since the light which forms the poles of the image is that which is not separated into an ordinary and extraordinary ray, and therefore passes unaltered through the crystal, then the light incident on the analyzer is exactly of the kind which the

analyzer can transmit to the eye, and therefore the pole is seen with full brightness. If it is the contrary, the light incident on the analyzer is exactly of that kind which is totally suppressed at the analyzer, and therefore the pole is perfectly black.

The third conclusion is that (supposing, to fix our ideas, that the direction of the light incident on the crystal is the same as that which the analyzer can transmit) the intensity of light depends only on the gain or loss of the ordinary or the extraordinary ray: being at its maximum when that gain or loss is a whole multiple of λ, and nothing when the gain or loss is an odd multiple of $\frac{\lambda}{2}$. For the first of these propositions it is only necessary to state that the crystalline plate, having resolved the incident light into two waves consisting of vibrations of different kinds, and having retarded one set more than the other by a whole multiple of λ, unites the two sets again in exactly the same condition in which they were at the resolution, and therefore they emerge forming a kind of light which is exactly similar to the incident light, and which is on that account susceptible of perfect transmission by the analyzer. This applies to quartz as well as to other crystals. For the second proposition we have only to remark that when circularly polarized light is incident on Iceland spar, nitre, and similar uniaxal and biaxal crystals, whatever be the position of the planes of polarization, it is resolved into two sets of plane vibrations at right angles to each other, one of which is $\frac{\lambda}{4}$ behind the other: and that when the relative path of these waves is altered by an odd multiple of $\frac{\lambda}{2}$, that which preceded by $\frac{\lambda}{4}$ now follows by $\frac{\lambda}{4}$ (neglecting multiples of λ): and the direction of the circularly-polarized light is thus reversed: and the emergent

light is therefore exactly of that kind which cannot be transmitted by the analyzer, or the corresponding point of the rings is black. This evidently will not apply strictly to quartz: but it will apply with tolerable accuracy when the rays make a considerable angle with the axis of the crystal.

The properties of Fresnel's rhomb suggest at once a method of constructing such an analyzer as we require. It is well known that if circularly-polarized light is incident on Fresnel's rhomb, it emerges plane-polarized, and the position of the plane of polarization at emergence makes an angle of + 45° or − 45° with the plane of reflection according as the incident light was right-handed or left-handed. Let the light emerging from the rhomb be received on an unsilvered glass at the polarizing angle, whose plane of reflection makes the angle + 45° with that of the rhomb. Now it is plain that if the light incident on the rhomb was right-handed, it becomes plane-polarized in the plane of reflection of the glass, and therefore is wholly reflected: if it was left-handed, it becomes plane-polarized in the plane perpendicular to the plane of reflection of the glass, and therefore is wholly suppressed. This combination therefore (a rhomb and an unsilvered glass at + 45°) has the property of wholly transmitting right-handed circular light and of wholly suppressing left-handed circular light: and in the same way it would appear that the combination of a rhomb and an unsilvered glass at − 45° has the property of wholly suppressing right-handed circular light and wholly transmitting left-handed circular light. Since all polarized light (and therefore light in general) may be represented by two pencils of opposite circularly-polarized light, it follows that our combination will have the power of resolving all light into two such pencils and suppressing one of them.

In practice, the use of Fresnel's rhomb in this part of the apparatus would (on account of its length) be attended with some inconvenience. I have therefore preferred for this purpose a plate of mica, of such a thickness that the ray polarized in the plane of one of its principal sections is retarded either $\frac{1}{4}$, $\frac{3}{4}$, or $\frac{5}{4}$ of a wave (according to the convenience of splitting) more than that polarized in the plane of the other. The mica being attached to the unsilvered glass so that its principal section makes an angle of 45° with the plane of reflection, an analyzer is produced which answers the same purposes, in general, as that described above. In strictness its effects are not the same, as the order of the colours is in some cases sensibly disturbed.

Upon trying this, when the incident light is circularly polarized, the general effects are precisely such as were anticipated. Iceland spar exhibits rings without a cross of any kind: nitre, arragonite, &c., exhibit the lemniscates uninterrupted in their whole extent, and without any trace of hyperbolic brushes. Unannealed glass exhibits dark patches surrounded by colours of different orders, without any continuous brush. In these and in other cases that I have tried, no alteration is produced in the appearances by turning the crystal, except that the system of rings, &c. is equally turned.

Perhaps the method of analyzation which I have described may, with the application of circularly-polarized light incident, be advantageously used for examining the nature of irregularly crystalline bodies. For instance: the appearances presented by unannealed glass in the common apparatus are singularly complicated: but in this they are comparatively simple. In this the eye sees at one glance (by the order of colour) how much the

ray polarized in one plane is more retarded than the ray polarized in the plane at right angles to it. The position of these planes however is not defined: that will be best found by observing the dark brushes in the common apparatus.

To give a mathematical form to the investigation, let us distinguish the two kinds of circularly polarized light by the letters A and B. Suppose that light of the kind A is incident, and that its vibration is resolved into $+a.\sin\frac{2\pi}{\lambda}(vt-x)$ for the ordinary ray, and $+a.\cos\frac{2\pi}{\lambda}(vt-x)$ for the extraordinary. On emerging from the crystal they may be represented by $+a.\sin\frac{2\pi}{\lambda}(vt-x)$ for the ordinary ray, and $+a.\cos\frac{2\pi}{\lambda}(vt-x+\Theta)$ for the extraordinary. These will be separated by the analyzer into two pencils of light of the kinds A and B; these will be represented by

$$\left.\begin{array}{ll} +p.\sin\frac{2\pi}{\lambda}(vt-x+q) & \text{perpendicular to the principal plane} \\ +p.\cos\frac{2\pi}{\lambda}(vt-x+q) & \text{parallel to the principal plane} \end{array}\right\} \text{constituting light } A,$$

and

$$\left.\begin{array}{ll} +p'.\sin\frac{2\pi}{\lambda}(vt-x+q') & \text{perpendicular to the principal plane} \\ -p'.\cos\frac{2\pi}{\lambda}(vt-x+q') & \text{parallel to the principal plane} \end{array}\right\} \text{constituting light } B.$$

Making the vibrations in each plane equal to those at emergence from the crystal,

$$a.\sin\frac{2\pi}{\lambda}(vt-x) = p.\sin\frac{2\pi}{\lambda}(vt-x+q) + p'.\sin\frac{2\pi}{\lambda}(vt-x+q')$$

$$a.\cos\frac{2\pi}{\lambda}(vt-x+\Theta) = p.\cos\frac{2\pi}{\lambda}(vt-x+q) - p'.\cos\frac{2\pi}{\lambda}(vt-x+q').$$

Or since this must be true for all values of $vt - x$, we may expand the sines and cosines and equate the coefficients of

$$\sin \frac{2\pi}{\lambda}(vt - x) \text{ and } \cos \frac{2\pi}{\lambda}(vt - x).$$

Thus we have

$$\begin{aligned} a &= p \cdot \cos q + p' \cos q' \\ 0 &= p \cdot \sin q + p' \sin q' \\ -a \sin \Theta &= -p \cdot \sin q + p' \sin q' \\ a \cos \Theta &= p \cdot \cos q - p' \cos q'. \end{aligned}$$

From these,

$$p = a \cos \frac{\Theta}{2}, \quad p' = a \sin \frac{\Theta}{2}, \quad q = \frac{\Theta}{2}, \quad q' = 270^\circ + \frac{\Theta}{2};$$

and the vibrations are

$$\left.\begin{aligned} &a \cos \frac{\Theta}{2} \cdot \sin \frac{2\pi}{\lambda}\left(vt - x + \frac{\Theta}{2}\right) \text{ perp. to princ. plane} \\ &a \cos \frac{\Theta}{2} \cdot \cos \frac{2\pi}{\lambda}\left(vt - x + \frac{\Theta}{2}\right) \text{ par. to princ. plane} \end{aligned}\right\} \text{constituting light } A,$$

$$\left.\begin{aligned} &a \sin \frac{\Theta}{2} \cdot \sin \frac{2\pi}{\lambda}\left(vt - x + 270^\circ + \frac{\Theta}{2}\right) \text{ perp. to princ. plane} \\ &-a \sin \frac{\Theta}{2} \cdot \cos \frac{2\pi}{\lambda}\left(vt - x + 270^\circ + \frac{\Theta}{2}\right) \text{ par. to princ. plane} \end{aligned}\right\} \text{constituting light } B.$$

If the analyzer is of such a kind that it can transmit light A, the intensity of the light that reaches the eye is $2a^2 \cos^2 \frac{\Theta}{2}$: if it transmits light B, the intensity of light is $2a^2 \sin^2 \frac{\Theta}{2}$. The former of these gives light at the place where there is no double refraction: the latter gives dark at the same place. But in both cases it is plain that the brightness depends only on Θ, the quantity which one ray has gained or lost on the other, and that it does not at all depend on the position of the planes of polari-

zation: and therefore in both cases the appearance will be that of patches or curves extending continuously through all the parts where the gain of one ray upon the other is a constant quantity.

In the former part of this paper I have alluded to an analyzer of a more general kind, namely one in which the light is separated into two elliptically-polarized rays. This is constructed by placing the plate of mica with its principal plane inclined to that of reflection at the unsilvered glass by an angle different from 45°. The investigation of its effects is not more difficult than that above, but is rather longer, and its results will hardly justify its insertion. I will only remark that, with the apparatus which I have supposed employed for the experiment above, if the Fresnel's rhomb or mica by which the incident light is made circularly-polarized, and the mica by which the new analyzation is effected, are turned the same way (leaving the glass reflector unmoved) the continuity of the rings is not interrupted, but a part of the image is seen to grow darker and darker; and when both are turned 45°, this dark part becomes the black brush. This supposes the planes of original polarization and of reflection at the glass to be at right angles; but if they are parallel the change is of the opposite kind, and the bright brushes are finally produced.

G. B. AIRY.

OBSERVATORY,
Jan. 19, 1832.

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XII. *On the Mechanism of the Larynx.*

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FELLOW OF CAIUS COLLEGE, AND OF THE PHILOSOPHICAL SOCIETY.

[Read *May* 18, 1829.]

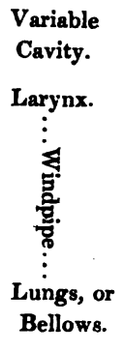
IT must be a source of great regret to those who have made themselves acquainted with the present state of Acoustics, to find the various investigations concerning the mechanism of the human voice leading to such unsatisfactory and even contradictory results. For from whence may more instructive lessons in that science be expected than from an apparatus which is capable of producing sounds in every variety of pitch, quality, and intensity, from the most exquisite music to the most execrable noise; an apparatus of no extraordinary dimensions, and one moreover of which the greater part is exposed to our observation during the various changes of form which it assumes whilst in action. Nevertheless the laws which connect these changes of form with the production and variation of the sounds are hitherto obscure. To account for this, we are compelled to refer to two considerations; on the one hand, the instrument of the voice is not exclusively appropriated to its production, but is also evidently adapted to the performance of functions far different and more important to the animal economy; and, on the other, the explanation of the phe-

nomena, in as much as they are produced by a part of the animal frame, has been consigned, with that of its other functions, to the Anatomist and Physiologist, to those whose professional studies are completely unconnected with Acoustics, a science which in all investigations of this kind must necessarily take the greatest share.

Accordingly every treatise on Physiology or Anatomy contains a chapter on the organs of voice, in which the parts conducing to its formation are described; whilst the contradictory and sometimes careless accounts which the best anatomical writers give of the mechanical action of these parts, and of the mode in which they perform their functions, form a vexatious contrast with the minute accuracy of their anatomical descriptions.

In the present memoir I have attempted a more minute analysis of a part of these organs than appears to have been hitherto undertaken. As, however, I am writing for philosophical readers in general, I have purposely divested my descriptions of the technical form as much as possible, and my drawings are to be regarded more as plans or types of the general structure gathered from the examination and comparison of many, than as representations of any one individual.

The vocal mechanism may be considered as consisting of ***Lungs*** or ***Bellows***, capable of transmitting by means of the connecting ***Windpipe*** a current of air through an apparatus contained in the upper part of the Windpipe, which is termed the ***Larynx***. This apparatus is capable of producing various musical notes which are heard after passing through a ***variable cavity***, consisting of the pharynx, mouth, and nose.



Now, if this arrangement be artificially imitated by combining together pipes and cavities with bellows in a similar order, and substituting for the Larynx any elastic lamina capable of producing musical notes when vibrated by the stream of air, it is found that by changing the form of the cavity above it, the various qualities which distinguish the continued notes of the human voice in speech, may be so nearly imparted to the sound which the imitative Larynx is producing, as plainly to shew that there is no necessity for seeking any power of altering the quality of the notes in the Larynx itself. This then may be considered as merely an instrument for producing certain musical notes, which are afterwards to be converted into vowels, liquids, &c. by the proper changes of form in the superior cavity.

We may here remark an essential difference between the vocal mechanism and our ordinary musical wind instruments, which are generally made up of some vibratory mouth-piece to generate the note, and an attached cavity, or pipe, to govern and augment its tone, each instrument having its peculiar quality; whereas the attached cavity in the vocal machine is capable not only of governing and improving the musical quality of the note, but also of imparting to it all manner of various qualities, the numerous vowels and liquids of speech, and also the perfect mimicry of the peculiar sounds of nearly all animals and musical instruments.

Of this cavity it is not my present purpose to speak. Indeed I doubt whether the science of Acoustics is sufficiently advanced to enable us completely to understand its mechanism. I shall in this memoir confine myself to the Larynx.

The precise form of the Laryngeal cavity, and the parts immediately connected with it, is shewn in Figs. 1 and 2. Fig. 2 is a section made by a plane passing through the nose, mouth,

and windpipe, called the mesial plane, dividing the head symmetrically, ***DEFG*** being the external outline of the throat, and ***H*** the back of the tongue. Fig. 1 is a section made by a plane perpendicular to the former, and passing along the line ***A·· B·· C.***

From ***A*** to ***D*** (Fig. 2) the windpipe presents a horizontal section nearly circular; above ***D*** it contracts in the transverse dimension, assuming the form of a narrow slit, termed the ***glottis.*** (The line ***GG*** (Fig. 1.) passes through the glottis). Immediately above it the windpipe expands into a pair of cavities, termed the ***ventricles*** of the Larynx, through which the line ***VV*** passes, and above these the passage again narrows into another slit indicated by the line ***LL,*** which has been termed the ***pseudo glottis.*** Above this the passage again expands, and finally opens into the ***pharynx,*** as the cavity behind the tongue is termed.

The whole surface of the cavity we have been describing is lined with a soft mucous membrane, similar to that which is seen on the inside of the mouth, soft palate, &c., with the exception of the edges of the glottis, where the lining assumes the form of a ligament, white, fibrous, and elastic, the outline of which is seen in Fig. 2, immediately below the opening of the ventricle. The edges of the pseudo glottis are formed merely by a kind of reduplication of the ordinary mucous membrane. It will be seen from Fig. 2, that neither the ligaments nor the ventricles extend entirely across the passage.

The most generally received opinion, and that which appears to me to be borne out by a careful investigation of the structure of the Larynx, is that the current of air from the lungs excites these ligaments to vibration, and so produces the sounds of the voice (vide note ***A***). Hence they are denominated the ***vocal liga-***

ments. I shall now proceed to a more minute examination of the precise nature of this vibration, and of the mechanism of the Larynx generally than appears to have been hitherto attempted. Assuming then that the source of the notes of the voice is to be found in the vibrations of a pair of membranous elastic edges, between which a current of air is allowed to pass, I shall endeavour to shew under what conditions such elastic edges must be presented to a current of air, in order that it may elicit from them the required vibrations.

One of the most commodious ways of investigating this is to prepare a piece of wood of the form *ABCD* (Fig. 9) and paste on one side of it a piece of fine kid leather, the upper end of which (*mn*) is cut straight, and a moderate degree of tension given to the leather when pasted on.

This arrangement presents us with an elastic membrane, whose upper edge *mn* is free and the other edges confined, and, therefore, with a case analogous to that of the vocal ligaments.

EF (Fig. 10) is the plan of a flat board, having a rectangular opening *GH* in the middle, and *LM* (Fig. 12) is a vertical section of this board along the line *IK*, (Fig. 10,) shewing a pipe *N* attached to the lower side of the board, in order to connect it with a pair of organ bellows, by means of which a current of air may be maintained through the rectangular opening at pleasure. (In Figs. 14, 16, and 18 are similar sections).

Let now an upright board *OP*, (Fig. 12,) be clamped upon *LM*, so that its face *P* may coincide with the side of the opening throughout its whole length. If the leather in its frame, (Fig. 9,) be exposed to the action of the current by placing the lower edge *BC* of its frame in contact with this board and always parallel to the side of the opening, and if then the frame be turned on

its lower edge, so as to place the plane of the leather at different angles with that of the board, the following phenomena will be observed:

If the leather be inclined to the board, as in Fig. 12, the current will merely drive it outwards and fix it with its upper edge concave to the board, as in Fig. 11, (which is a bird's-eye view of the upper edge).

If the leather be inclined from the board, as in Fig. 14, the current will draw the upper edge inwards, maintaining it with its upper edge convex to the board, as in Fig. 13.

If, however, the leather be placed in the intermediate position to these two, that is, parallel or nearly so to the board, the current will excite and maintain strong vibrations in the upper edge of the leather, producing a loud musical note, as long as the current is kept up.

If for the board $^{OP}_{\wedge}$ we substitute a similar frame with leather, and apply the two frames opposite to each other, above the rectangular opening of the board, as in Figs. 16 and 18, we have an arrangement somewhat resembling the glottis, in possessing a pair of edges opposite to each other; with this, similar phenomena to those just described are observable; namely, when the leathers are inclined *to* each other, as in Fig. 16, the current maintains their upper edges in the position Fig. 15. When they are inclined *from* each other, as in Fig. 18, the current fixes them in the position Fig. 17, but when nearly parallel puts them into strong vibration. The angle at which they may be inclined to each other to make vibrations possible varies with the tension of the leather and the force of the current. For an examination of the reasons of these phenomena I must refer to Note *B*; it is sufficient for our present purpose to know that it is not merely necessary for the vibration

of a pair of ligaments, like those constituting the glottis, that a current of air be passed between them, but that their opposite surfaces must also be placed in a given position with respect to each other.

For instance, Fig. 5 is the ordinary position of the ligaments *GG*, in which the breath passing between them could never produce a sound from them, they being inclined from each other. Whereas, in Fig. 1, where they are parallel, the breath would instantly excite vibrations in them.

Here we have a solution of a difficulty which never seems to have occurred to former writers; that is, why the ligaments are silent while the ordinary breath passes between them. It cannot be because their tension is not sufficient, because I shall shew that they are always in a state of tension nearly corresponding to the pitch of the speaking voice.

To shew how the same pair of ligaments may produce various notes, let a wooden pipe be prepared of the form Fig. 19, having a foot *C* like that of an organ pipe and an upper opening, long and narrow as at *B*, with a point *A* rising at one end of it. If a piece of leather, (or, still better, of Hancock's sheet India rubber,) be doubled round this point and secured by being bound round the pipe at *D* with strong thread, as in Fig. 20, it will give us an artificial glottis with its upper edges *GH*, which will vibrate or not, at pleasure by inclining the planes of the edges, according to the previous experiments. A couple of pieces of cork *EF* may be glued to the corners to make them more manageable. From this machine various notes may be obtained by stretching the edges in the direction of their length *GH*; the notes rising in pitch with the increased tension although the length of the vibrating edge is increased. It is true that a scale of notes of

equal extent to that of the human voice cannot be obtained from edges of leather, but this scale is much greater in India rubber than in leather, and the elasticity of them both is so greatly inferior to that of the vocal ligaments, that we may readily infer that the great scale of the latter is due to its greater elastic powers. To obtain various notes from the glottis, therefore, it is only necessary to vary its longitudinal tension after its ligaments have been placed in the proper position.

As, however, during breathing the air passes freely in and out of the lungs through the identical apparatus by which the notes are produced, the passage we have been considering, or glottis, must be capable of assuming the form of a large and free aperture; since it is certain, from the freedom with which the air is inhaled and exhaled, that it is not compelled to pass through so small a slit as the glottis appears to be during vocalisation.

The passage is also capable of being shut so close by its own small muscles that all the exertions of the powerful abdominal muscles acting upon the diaphragm to compress the lungs and condense the air in the trachea are not capable of forcing it open.

The production of a musical note takes place instantaneously, at the pleasure of the individual. The breath has been previously traversing the passage in silence, and at our will some change is immediately made in the larynx, which produces the note, and this certainly depends upon something more than the mere closing of the passage, because we can make the aperture of the passage pass through all degrees of contraction up to absolute closing during the expiration of the breath without producing any sound, except the usual rushing noise of a forcible current of air passing through a narrow aperture.

The law of vibration which I have above explained renders this more intelligible; for since it appears that unless the membranous edges of the passage are placed nearly parallel they cannot be made to vibrate, we have only to suppose that the *change we feel in the Larynx* is the placing of the ligaments in a parallel position, and the whole mystery is explained. If therefore I can succeed in shewing that the arrangement of the cartilages and muscles is adapted for the purpose of placing the vocal ligaments under the various conditions which have been shewn to be necessary, I shall have done all that is possible to complete the evidence in favour of my explanation.

For, after all, no explanation of the functions of a machine, of which essential parts are concealed while in action, can be complete and uncontrovertible: when we have examined the separate parts, and have enumerated the functions which observation shews the machine to be capable of performing, and by comparing these with the different portions here elicited, as we flatter ourselves, a complete allotment of each function to its appropriate part of the structure, we have only been in fact describing a machine of our own contrivance, copied in form, and capable perhaps of performing the same functions, but not necessarily identical with the original, because we cannot certainly know whether it performs the same motions for the same functions. Hence we can only establish a probability that the uses of the corresponding parts in the two are the same. Beyond this probability we can never get, unless we can succeed in viewing the machine in motion.

This is not entirely the case with the Larynx, because we are enabled to trace the motions of some of the cartilages from without; but the greater part of the machine is, and always must be, hidden

from our view while living, for we cannot make much use of the facts said to have been observed by some Physiologists in their experiments on living animals, which are reported by men plainly but loosely acquainted with Acoustics, and which, as they have been deduced from a vocal mechanism vastly inferior to the human, may very probably mislead us if we attempt to apply them to the explanation of the latter.

Having now, in some degree, considered the uses of the Larynx, and laid down some principles, we may proceed to examine its structure with more minuteness.

Upon removing the mucous membrane which lines the whole of the interior of the laryngeal cavity, but leaving the vocal ligaments in their place, we find the latter supported in a curious frame of cartilages united by certain ligaments and articulations, and provided with muscles, by the action of which the cartilages may be made to assume various positions with respect to each other, and thereby alter the tension and relative position of the vocal ligaments.

The windpipe is found to consist of a pile of cartilaginous rings, serving to keep the passage from the lungs always open, and forming in this respect a contrast with the œsophagus, or tube leading from the cavity of the pharynx to the stomach, which is always closed by its muscular contractile structure, excepting at the moment of the passage of food. In investigations concerning the organs of voice the œsophagus may always be regarded as having no existence. Its place is indicated in Fig. 2 by the line *IK*.

Above the rings of the windpipe, however, is a stout bony annulus, denominated the cricoïd cartilage, which serves as the foundation of the mechanism we are about to describe.

Fig. 3 is a section of the Larynx similar to Fig. 2, but representing it as stripped of its mucous membrane, &c., leaving the bare cartilages with the vocal ligament in its proper place, and also the muscles.

Fig. 4 is an external view of the corresponding half of the Larynx, and Fig. 7 a bird's-eye view of the entire Larynx, both in the same state of dissection. Fig. 8 is an enlarged sketch of part of the upper half of Fig. 7. In these four figures the same parts are indicated by the same letters.

The cricoïd cartilage is seen (Figs. 3, 4, *ABC*,) surmounting the rings of the windpipe. The thyroïd cartilage *ECGH* embraces the cricoïd, and is articulated to its sides by its lower horns at *C*, so that it may be regarded as turning round the point *C* as a fulcrum. As this discussion merely regards the motion of the cartilages among themselves, it is of no consequence whether we regard the cricoïd or thyroïd as fixed, and for convenience I shall assume the cricoïd as fixed for the present.

Upon the upper surface of the back part of the cricoïd are seated two small cartilages (*FF*, Figs. 7, 8,) termed the arytenoïds. They are placed upon articulating surfaces which are formed on the upper outer edge of the cricoïd, and which may be considered as portions of cylinders, whose axes are inclined, both with respect to the horizontal and vertical sections. In the vertical section the projection of this articulating axis is in the position *BL*, Fig. 3, and in the horizontal in the line *O P*, Fig. 8. The base of each arytenoïd is spread out, and curved below *Q*, Fig. 8, so as to lie upon this articulating surface, to which however it is so loosely adapted as to permit a small degree of sliding motion transverse to the axis. The arytenoïds are however firmly tied to the back of the cricoïd by a bundle of strong ligaments *BR BS*, Fig. 8,

and *BF*, Fig. 3, diverging from the point *B*, Fig. 8, which point *B* is as nearly as possible the point where the axis of the articulating surface would intersect the cricoïd. The vocal ligament is stretched from the thyroïd at *T* to the arytenoïd at *V*, and as there is no muscle which can relax the ligament *BRS**, it receives and transmits to the point *B* of the cricoïd the tension of the vocal ligaments.

The motion of the arytenoïd is therefore compounded of a rotation round the axis *OP*, and of a sliding motion transverse to this axis, which is confined by the tension of *BRS* to a swinging round the point *B*, of which we shall presently see the use.

We have already seen that the thyroïd is so united to the cricoïd as to turn round the point *C*, Figs. 3 and 4, as a fulcrum. The effect of this rotation is to alter the distance between the point *E* of the thyroïd, and *B* of the cricoïd, and therefore to affect the tension of the vocal ligaments.

If this distance be increased by the thyroïd revolving in the direction from *B* to *E*, the tension of the vocal ligament is increased, and by its pulling at the arytenoïd cartilage the tension of the bundle of ligaments is increased. If the distance *EB* be diminished by the thyroïd turning in the reverse direction, the contrary effect will take place.

To produce this motion, two pair of muscles are provided, one of the external pair (the cricothyroïd muscle) is seen at *AK*, Fig. 4†, when this muscle contracts it brings the point *K* of the

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* Vide Note *C*.

† Each muscle of this pair is sometimes seen divided into two, and is described by some writers as such. Some of the fibres are attached so close to the fulcrum *C* as to be apparently intended to stretch the ligaments which bind the horns *C* of the thyroïd to the cricoïd, and thereby unite more firmly these two cartilages during vocalisation.

thyroïd nearer to the point *A* of the cricoïd, and therefore increases the distance *EB*; *this pair of muscles therefore stretches the vocal ligaments.*

One of the internal pair (the thyroarytenoïd muscle) is seen at *Emae*, Fig. 3; it is attached to the inside of the front of the thyroïd at *Em*, and to the arytenoïd at *ae*; when this muscle contracts it approximates the arytenoïd to the point *E*, and as the arytenoïd is tied to the cricoïd by the bundle of ligaments at *B*, it of course draws the point *B* after it, just as if the muscle were attached immediately to *B*. The effects of this muscle is then to decrease the distance *EB*, and therefore to *relax the vocal ligament*.

Hence the thyroarytenoïd muscle is the antagonist muscle of the cricothyroïd, and together they govern the pitch of the notes.

The truth of this account of the stretching and relaxing of the vocal ligaments may easily be verified, as far as the motion of the cartilages is concerned, by a method which was first suggested by Ferrein*, but appears to have been forgotten or misunderstood by succeeding writers. We may readily trace with the finger on the outside of the throat, (at *GFED*, Fig. 2,) the thyroïd cartilage *EF*, the cricoïd cartilage *DM*, and a small space *ED* between them (marked *mn* in Figs. 3 and 4). Now it is plain that when the thyroïd revolves upon *C* in the direction *BE*, so as to stretch the vocal ligaments and raise the pitch of the notes produced, that this motion approximates the lower edge *m* of the thyroïd to the upper edge *n* of the cricoïd, and, therefore, diminishes the aperture *mn*, and vice versâ, when the ligaments are relaxed, the aperture *mn* is increased.

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* Ac. Par. 1741.

But upon singing a scale of notes two motions are to be observed in these cartilages; one is a general motion upwards, when the pitch of the notes rises, and downwards, when it falls; which we have no concern with at present, as we are treating only of the motions of the laryngeal cartilages with respect to each other, which this does not affect. The other motion consists of the relative motion of the cricoïd *D* and the point *E* of the thyroïd, and consequent variation in the distance *DE*, which is best to be traced by lodging the tip of the finger in the little hollow between the cartilages, and so following the general motion up and down. By doing this carefully, the size of this aperture will be perceived to follow a law exactly coinciding with the above explanation, namely, always increasing with a descending pitch and diminishing with a rising one.*

So far, the arytenoïds have merely served as links, connecting the vocal ligaments and thyroarytenoïd muscles with the cricoïd cartilage, and the function just described would be just as well performed if the vocal ligaments and thyroarytenoïd muscles were attached to the cricoïd at *B* without their intervention. In fact, were the vocal ligaments merely intended to sound whenever the current of air passed through the larynx these cartilages would apparently have no office.

But it is to be remembered, that to enable the ligaments to vibrate they must be made to assume a peculiar position with respect to each other, and that for breathing, it is necessary that we have the means of opening the passage wide, also of entirely closing it; during which it is essential that that peculiar position be avoided, for fear of a sound being produced when not intended. It is in the performance of all these motions that the arytenoïds are concerned.

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* Vide Note *D*.

The articulation of the arytenoïds with the cricoïd has been already described. From the extremity *N*, (Fig. 8,) of the arytenoïd arises a muscle, termed the *cricoarytenoïdeus posticus*, which is turned round the edge of the cricoïd, and affixed to the lower part of the back of the latter cartilage. Its mechanical action, however, is the same as if it acted on the arm of a short lever *N*, in the direction *NW* on the plan, that is to say, perpendicular to the axis of motion *OP*, and its effect is to produce rotation about this axis, and therefore to separate the arytenoïd cartilages from each other and open the passage. (Vide Note *C*.)

From the arytenoïd another muscle *NX* arises, and is attached to the cricoïd at and about the point *X*; this is termed *cricoarytenoïdeus lateralis*. In Fig. 3, the fibres of this muscle may be seen arising from *X* and passing up to the arytenoïd, lying nearly parallel to the projection of the axis of motion.

To understand the action of this muscle upon the arytenoïd, we must remember that the latter is attached to the point *B* by ligaments, which radiate from this point, and are united to the arytenoïd along its posterior surface from *S* to *R*.

The tension of this muscle then in the direction *NX* will, by drawing the cartilage in that direction, stretch the ligament *RB*, and tend to bring the points *XNB* into the same straight line; this will at the same time approximate the point *V* to the medial plane and corresponding point of the other arytenoïd, and also, (as it appears from Fig. 3 that in the vertical projection, *N* is above the line joining *BX*,) it will depress *N* and still more *V*, because the cartilage turns on the articulating surface beneath *Q*.

The effect, in short, of the pair of muscles in question is to press the points *V* of the arytenoïds together, at the same time depressing them.

The two arytenoïds are moreover united by a muscle, called the *transversus arytenoïdeus*, which arises from *R d* of one arytenoïd, and is united to the other in the corresponding points. Its section is seen at *s* in Figs. 2 and 3. It is removed in all the other figures. Its action upon the arytenoïds is plainly to press together the point *S* and its corresponding one. Hence, when this muscle acts at the same moment with the cricoarytenoïdei laterales, which we have been just considering, their joint effect will press the whole of the adjacent faces of the arytenoïds together, depressing the points *V* and closing the glottis, and, therefore, antagonizing the action of the cricoarytenoïdeus posticus. (Vide Note *E*.)

Indeed, it appears at once from the diagram that the forces *NX NY* of these two muscles must together produce a resultant in the direction nearly of *WN*, and therefore directly opposed to the action of the cricoarytenoïdeus, which is represented in direction by *NW*.

Hence, the cricoarytenoïdei postici open the glottis. The cricoarytenoïdei laterales and the arytenoïdeus transversus acting jointly close the glottis.

The complete closing of that portion of the aperture which is included between *T* and *V*, (Fig. 8,) appears to be effected jointly by the motion of the arytenoïd cartilages, which in closing together approach the point *T*, from the obliquity of their axis of motion, and by the swelling of the muscle *NX* in contracting to bring the arytenoïds in contact; both causes tending to compress the cellular tissue and muscular fibres which occupy the space *TXNV*, and therefore to close tightly together the sides of the passage below the vocal ligaments.

We have now to consider the means by which the vocal ligaments are placed in the proper relative position for vibration. To explain this, let (Fig. 6,) the continued line be the ordinary position of the

glottis for breathing, when it is slightly opened. In this position the vocal ligaments *ab cd* diverge from each other in such a manner that, according to our previous experiments, the current of the breath could never excite them to vibration, whatever their longitudinal tension might be. If the points *ac* be carried upwards, at the same time approaching each other, so as to acquire the position *a'c'*, it is manifest that this change, by increasing the distances *ea*, *fc*, will, by diminishing the convexities *eba fdc*, draw the passage into the form indicated by the dotted lines *eba' fdc'*, in which the vocal ligaments have assumed the position proper for vibration. But the motion of the arytenoïd round the axis *OP*, (Fig. 8,) which we have already described, will, in raising the vocal ligaments, separate them and take them rather into the position indicated by the dotted line *ea''*.

It only remains then to explain how the extremity *V*, (Fig 8,) of the arytenoïd may be made to rise and approach the corresponding point of the other arytenoïd at the same time; for, if this is done, the vocal ligaments will necessarily assume the required position.

This motion is permitted by the sliding of the articulating surface of the arytenoïd upon the cricoïd, already described, and is effected by the thyroarytenoïdei muscles. These muscles we have shewn to be only employed during vocalisation, and we shall now see that their peculiar structure places at the same time the arytenoïds in the proper position for vibration.

The internal face of one of these muscles is seen in Fig. 3; a bird's-eye view of the opposite one is shewn in the lower half of Fig. 7 at *kVf*. That corresponding to Fig. 3 is removed from the upper half of Fig. 7, to shew the cricoarytenoïdeus lateralis *NX* more distinctly, for a similar reason the latter muscle is removed from the lower half to display the thyroarytenoïdeus *kVf*.

In Fig. 3 the arytenoïd and its attached vocal ligament and muscle are in the vibrating position. When the arytenoïd is in the position corresponding to Fig. 5, the point *F* is considerably below the line *EB*. Hence when the thyroarytenoïd muscle is brought into action, its fibres, which lie on this face parallel to the vocal ligament, tend of course to bring the points *EFB* into a straight line and hence raise the point *F*. This, by inducing a rotation round the axis *OP*, would separate the ligaments were it not counteracted by the direction of the fibres of the lower portion of the muscle, which arising from about *m*, near the median plane are attached to the arytenoïd at a much greater distance from it; this may be seen clearly in Fig. 7. They, therefore, draw the point *N* (Fig. 7, 8) of the arytenoïd towards the median plane producing the sliding motion so often alluded to, whilst at the same time the upper fibres of the muscle, which are not parallel to the lower, maintain the upper part of the cartilages in their due position with their points separated, so as to part the upper ligaments of the glottis and keep them out of the way of the current of air.

This may be elucidated by considering the muscle when in the position corresponding to Fig. 5 as a very loosely twisted rope, which when brought into action tends by untwisting itself to bring its fibres into parallelism, and therefore to communicate a rotatory motion to the attached arytenoïd, which, combined with its articulation to the cricoïd, places it in the exact position required for the vibration of its vocal ligament, which then assumes the form Fig. 1.

As the arytenoïds are hidden from our sight, and cannot be traced externally as the other cartilages can, it is plain that the whole account I have given of their motions must be considered as depending entirely upon induction.

With respect to the scale of notes in the human voice, which is termed the falsetto, I shall merely observe that it is at present extremely doubtful whether it owes its peculiar quality to some change in the laryngeal mechanism, or in the superior cavity; the motion of the cartilages observed by the finger from without shews that the tension and consequent diminution of the aperture *ED*, Fig. 2, goes on in the production of these notes just as it does in that of the natural tones, and is therefore carried so far in the higher notes of the falsetto that the space *ED* is completely obliterated by the upper edge of the cricoïd touching the lower border of the thyroïd.

According to M. Magendie* the vocal ligaments of a dog vibrate through their whole length while producing deep notes, but in high notes the hinder portion only vibrates, the thyroïdean extremities being closed together so as to shorten the aperture of the glottis, this diminution of the glottis becoming greater and greater as the notes rise in pitch. Should this ever be established to be the case in man, I should not be surprised if it were found that only during the production of the natural scale the vocal ligaments vibrate through their whole length, after the manner I have described already; while for the production of the falsetto notes the following changes may be introduced. If the arytenoïds be pressed together by the arytenoïdei transversi and cricoarytenoïdei laterales, and at the same time lifted up into the vocal position by the thyreoarytenoïdei†, the complete closing of the passage will be prevented, and notes will be produced by the action of

* T. I. p. 215.

† The cricoarytenoïdei laterales press the points *V* together (Fig. 8), at the same time depressing them; but the thyroarytenoïdei approximate the points *V*, at the same time raising them, and without bringing them into contact. If both these muscles act at once the raising effect of the latter is greater than the depressing effect of the former, because the latter acts

the current which will differ entirely in their quality from the former, because the vibrating length of the ligaments will be diminished by the contact of the arytenoïds, and they will beat against each other during vibration.

The approach of the cricoïd *D* to the thyroïd at *E*, Fig. 2, will however compress the cellular tissue, &c., and tend to press together the vocal ligaments and the sides of the passage below them, beginning at the thyroïdean extremity of the glottis, and diminishing the vibrating portion at that extremity; and if the tension of the ligaments be increased this compression will finally close the aperture in the manner described by M. Magendie.

I have been more minute in examining the mechanical action of the muscles than may at first sight have seemed necessary, because for want of some such examination the greatest confusion prevails in all the accounts of them. Thus, while all writers agree that the cricothyroïdei serve to approximate the cricoïd cartilage to the thyroïd, either by raising the cricoïd or depressing the thyroïd, none of them have shewn how these cartilages are to be separated again, neither do they agree as to the effect of this approximation upon the glottis. Again, Cowper and Albinus make the thyroarytenoïdei draw the arytenoïds nearer together; but Sömmerring and Haller make them separate these cartilages; and Meckel and others make them draw forward the arytenoïds. Haller thinks that they relax the vocal ligaments; Bichat that they stretch them. The cricoarytenoïdei laterales are stated by Cowper, Haller, and

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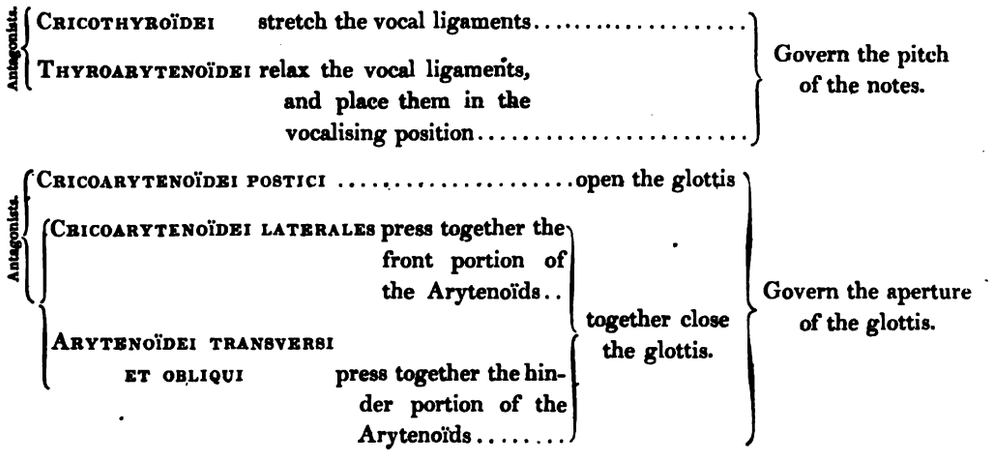
on a longer lever than the former: but, on the other hand, the mechanical action of the former muscles to press the points *V* strongly into contact, by bringing *XNB* into a straight line, receives very weak opposition from the latter muscles, which have already brought *ghB* into a straight line; therefore the joint effect of these two pair of muscles will be to press together and raise the points *V*.

Magendie, to open the glottis by separating the arytenoïds, but by Sömmering and Bichat to close it. Other writers follow one or other of these opinions, combining the different muscles after their own fashions, without attempting to support their statements by mechanical reasoning deduced from the structure and connexion of the parts. In the following table I have brought together the functions of the muscles according to the views I have taken in the preceding pages.



Lest it should appear to some of my readers that I have, in stating the uses and actions of the several parts of the Larynx, expressed myself more decidedly than I ought to have done, I beg to state that this decided style was adopted for the sake of brevity, and that it is with the greatest deference that I have ventured to offer opinions in many cases so different from those of former writers. At the same time I have endeavoured to explain the mechanical grounds of these opinions as clearly as possible, and to distinguish carefully between those portions of my explanation that rest on mechanical facts, and those that are merely deduced from inductive reasoning.

ROBERT WILLIS.

NOTES.

Note A.—Page 326.

M. Savart, whose labours in every branch of Acoustics have contributed so greatly to the advancement of that science, has written an ingenious Memoir, (Annales de Chimie, t. 30), in which he has endeavoured to shew that the sounds of the Larynx are produced, not by the vibration of the vocal ligaments, but in a manner analogous to those of the little instrument called a duck whistle, of which he has given a theory with experiments; this machine consists of a small circular box, in the centers of the flat sides of which are two holes exactly opposite to each other. When a current of air passes through these holes a sound is produced, and his whole explanation rests upon the analogy between the section of the Larynx (Fig. 1.) and of this instrument, of which the glottis and pseudo-glottis are supposed to represent the two holes, and the ventricles the cavity. But his mode of obtaining the form of the laryngeal cavity is to take a cast of it in plaster, by which the ventricles are of course distended, and made to assume a magnitude and consequence which they never can possess during life, but which are essential to his theory. Neither does it appear to me that he has been successful in applying this explanation to the muscular structure of the Larynx. This instrument had been before made use of with great success by Kempelen for the explanation and imitation of the whistling and hissing sounds of the human voice, (Vide *Mech. de la Parole*).

These ventricles appear to have no use considered as cavities, but to arise merely from the form of the lining of the Larynx, which, after separating above the glottis to isolate the vocal ligaments, and leave them free for vibration, again returns to form the pair of folds which constitute the pseudo-glottis, and serve to protect the glottis from the accidental intrusion of foreign bodies.

The most ordinary appellation of the vocal ligaments is *vocal chords*, but this term, which implies an *isolated* vibrating ligament, ought certainly to be abandoned as conveying a most erroneous notion of the structure of the parts in question. They have also been termed the *lips of the glottis*.

Note B.—Page 328.

Let *CD*, *AB*, Fig. 22, be the longitudinal section of a tube, the transverse section of which is a parallelogram, whose longest side is considerably greater than its shortest, which is equal to *AC*.

Let this tube be terminated on its upper side by an elastic membrane *DE*, attached on three sides to the tube, but having a free edge opposite *B*, (similar to the membrane in Fig. 9).

Suppose the extreme position of this membrane in performing vibrations to be *DF* and *DG*, and let a current of air be passing along the tube in the direction of the arrow. Now when a membrane vibrates under these circumstances its motion will be influenced by two causes.

First,—It is well known that when a current of air passes through a diverging tube, such as *ACDGB*, that, by what is called the lateral communication of motion, it gradually communicates its onward motion to the particles of air which were at rest in *EDG*, and carries them away with it creating a rarefaction in *EDG*, which occasions a superabundant pressure on the outer surface of *DG*, by which it will be urged towards *DE*. This therefore acts as a retarding force when the membrane is passing in the direction *EG*, and as an accelerating force when it is moving in the opposite direction.

Second,—In a tube of the form *ACDFB* the current exerts an outward pressure, which will act on the membrane *DF*, accelerating it in its passage from *F* towards *E*, and retarding it during its return to *F*. There is an intermediate position *DH*, in which the current exerts no pressure on the sides of the tube, and therefore none on the membrane.

We may easily conceive then that when the current was first admitted into the tube, it might, from the first cause, occasion a superabundant pressure upon the outer surface of *DE*, which would set it in motion towards *DF*. By virtue of this motion it would pass the line of equilibrium *DH*, and would then be soon brought to rest by the resistance arising from the second cause and its own elastic force. From this position its elasticity and the pressure of the second cause would return it, it would pass *DH*, be again brought to rest by the resistance of the first cause, again return, and so on; in this way it would oscillate for some time till the friction and resistance of the air, rigidity of the membrane, &c. gradually reducing the extent of its vibrations, would bring it to rest in the position *DH*.

Experiment shews however on the contrary that as long as the current is maintained the vibrations continue. This may perhaps be explained by a closer examination of the nature of the force arising from the lateral communication of motion. When the membrane is passing from *DF* towards *DG*, and has got into the position *DH*, we suppose this phenomenon to begin; but as the rarefaction it occasions proceeds from a motion gradually imparted to the air, it is plain that it takes *time* to perfect it, and hence at any given point *K* the rarefaction is not so great when the membrane passes it in going towards *DG*, as it is when the membrane returns from *DG*.* Hence the retarding force at each point *K* in going

* This effect is assisted too by the circumstance that when the membrane is receding from the current, the circumambient air can more easily rush in to supply the deficiency than it can when the membrane is returning.

outwards is less than the accelerating force at the same point in returning, and the difference gives us a force to balance the loss from friction and resistance, which will therefore keep the membrane in motion as long as the current is kept up.

A similar explanation will apply to the case of the reed of an organ pipe, to the free reeds now so much in vogue, and to every other case in which a vibratory motion is maintained by a current. For instance, let *ABCD* Fig. 21, be a transverse section of the plate of a free reed, and let *EF* be the two extreme positions of the vibrating tongue which passes through the aperture *BC* of the plate, the dotted line being its position of rest. When in the position *F*, the current indicated by the arrows rarifies the air above the tongue by the lateral communication of motion, which action ceases the moment the plate gets to the level of *BC*, while a similar process commences at the lower surface of the tongue, and ceases when it returns to *BC*; affording in both cases a retarding force in going from *BC* less than the accelerating force in returning to it, and therefore maintaining the motion as long as the current is kept up. Here the tongue is first started into motion by the upward pressure of the current; and if the position of the tongue be not accurately adjusted, it is found that it will either assume a position of rest a little above that which it takes when no current acts upon it, or else will get very slowly into motion.

I propose to enter more fully however into this subject hereafter.

M. Biot* has attempted to explain the motion of an organ reed in a way which would be perfectly satisfactory upon the hypothesis of perfect elasticity and non-resistance of the air, but in no other case. Were his

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* Physique, t. II. pp. 166, 172. Precis elementaire, t. I. pp. 429, 431. Also Pouillet. Physique, t. II. p. 180.

M. Biot also adapted two lips of India rubber to a pipe connected with organ bellows, and upon passing the current of air through them he obtained sounds. (Precis elementaire de Physique, t. I. p. 462.)

view of the action of the current correct, it is manifest that the reed would, after a few oscillations, assume a position of rest in every case. These remarks of mine have suggested to Professor Airy the investigation of an elegant law, for which I must refer to his ingenious paper in the previous volume (p. 369) "On Certain Conditions under which a Perpetual Motion is possible."

Note C.—Page 337.

The offices generally assigned to the cricoarytenoïdeus posticus are to open the glottis by drawing the arytenoïd backward, and to stretch the vocal ligament. Now it is perfectly true that this muscle (*NW* Fig. 8) in drawing the arytenoïd from the mesial plane to open the glottis will affect the tension of the vocal ligament *TV* by increasing the distance *TV*. But the only function in which the tension of the ligament is concerned is vocalisation, and for this a peculiar position of the ligament is required, which is given by the thyroarytenoïdei, while the complete and direct regulation of the tension is also provided for by the joint action of the cricothyroïdei and thyroarytenoïdei. On the other hand, the effect of the cricoarytenoïdeus posticus upon the tension is very slight at the first departure of the point *V* from the mesial plane, indirect and inconsiderable in every case, and it cannot act without drawing the cartilages asunder and out of the vocalising position; therefore I infer, that this muscle is never concerned in adjusting the tension for vocalisation, and that its effect upon it may therefore be neglected.

Again, the phrase "drawing the arytenoïd backward," is a loose one, and implies that the ligament *BRBS* is relaxed by this action, which is by no means the case. I have attempted to shew that this muscle produces rotation round the axis *OP*; and as the bundle of ligaments radiate from about that point *B* of the cricoïd where the axis intersects its surface, it is plain that the rotation of the arytenoïd will scarcely affect their tension. It is true that those fibres of the muscle which lie nearest

the mesial plane are directed so as to draw the arytenoïd towards *B*; but this is counteracted by the fibres that lie farthest from the mesial plane; and as we may assume that the whole of the fibres of the muscle act at once, the resultant of their action will be found as nearly as possible perpendicular to the axis of articulation *OP*.

Note D.—Page 336.

It is worth while to ascertain the state of tension of the vocal ligaments when at rest, which we may readily infer from the application of the test here described. If the finger be lodged in the space *ED*, Fig. 2, and a bass note sounded, the larynx will descend from its position of rest, and *ED* be enlarged; if a high note be sounded, the larynx will ascend from its usual position, and the space *ED* be diminished; but an intermediate note may be found, the sounding of which will not remove the larynx from its ordinary position of rest, or alter the usual magnitude of *ED*, and this note will be the average pitch of ordinary speech. Now, as I have shewn that the space *ED* indicates the tension of the vocal ligaments, I infer from this that in the position of rest these ligaments possess the tension required for the average pitch of speech, requiring to be relaxed for deeper notes, and stretched for higher. But as in this state of tension the breath passes between them without being able to elicit vibrations from them, we see the necessity of some such conditions as those I have described to enable sounds to be produced at pleasure.

Note E.—Page 338.

Some of the fibres of the arytenoïdeus transversus are attached to the cricoïd at one extremity, and are sometimes described as distinct muscles under the name of arytenoïdei obliqui; they conspire with the transverse fibres in drawing together the hinder portion of the arytenoïds, and by their oblique direction assist the cricoarytenoïdei laterales in depressing the arytenoïds.

DESCRIPTION OF THE PLATES.

PLATE XXII.

THE letters of reference belonging to the shaded figures on this Plate are contained on its accompanying outline Plate, marked Plate 22*.

FIG. 2 is a section of the vocal mechanism in its natural state, made by a plane, technically called the mesial plane, passing through the mouth, larynx, &c. and dividing the head symmetrically. In the outline of this Figure the mouth, chin, tongue, &c. are sketched in to make the relative situation of the Larynx to these parts more clear; the shaded Figure is confined to the Larynx and parts immediately adjacent.

MPA, upper part of windpipe.

MTOLIP, laryngeal cavity, opening at *LI* into the pharynx.

trIL, part of the cavity of the pharynx, which is continued above *tr* into the nostrils.

efghHk, cavity of the mouth.

a, *b*, lips. *c*, *d*, teeth. *ef*, palate. *fg*, soft palate. *h*, uvula.

kH, tongue. *LO*, epiglottis. *IK*, œsophagus.

TV, vocal ligament, constituting one side or lip of the glottis.

TB, upper ligament, constituting one side of the pseudo-glottis; the dark opening between these is that of the ventricle.

PQ, *DM*, cut edges of cricoïd cartilage.

ETF, cut edge of thyroïd cartilage.

s, cut edge of arytenoïdeus transversus.

FIG. 1 is a section made by a plane perpendicular to the former, and passing along the line *AB* and *BC* in Fig. 2; looking towards *IK*.

The line *GG* passes through the vocal ligaments and glottis.

LL, through the superior ligaments and pseudo-glottis.

VV, through the two ventricles.

Had this section passed nearer to *EF* (Fig. 2) these ventricles would have appeared somewhat deeper, and considerably higher at their inward extremities.

Fig. 5 is a similar section, having the vocal ligaments in another position, and Fig. 6 an enlarged diagram of part of these sections, which is sufficiently explained in the text.

In Figs. 3, 4, 7, 8 the Larynx is represented as removed from the surrounding parts, and stripped of the epiglottis and investing mucous membrane, leaving the bare cartilages, muscles, and ligaments; but still retaining in their proper relative positions those parts which are left.

Fig. 3 is a section of the Larynx in this state, corresponding to Fig. 2; the cut edges of the cartilages being therefore alike in these two figures.

EmCG, the thyroïd cartilage. *G*, its upper horn. *C*, the place of its lower, by which it is articulated to the cricoïd.

AnBC, the cricoïd cartilage.

F, the arytenoïd cartilage.

EF, the vocal ligament.

FB, the bundle of ligaments uniting the arytenoïd to the point *B* of the cricoïd.

Emea, the thyroarytenoïdeus muscle.

Xe, the cricoarytenoïdeus lateralis.

s, the transverse section of the arytenoïdeus transversus.

BL, the projection of the axis of articulation of the arytenoïd with the cricoïd.

mn, the space between the thyroïd and cricoïd, which may be traced externally at *DE* (Fig. 2.)

Fig. 4, the external elevation of the half of the Larynx removed from Fig. 3.

EmeH, the thyroïd cartilage. *H*, its upper horn. *C*, its lower horn, articulated to the cricoïd.

AnBC, the cricoïd cartilage.

AK, the cricothyroïdeus muscle.

Fig. 7. A bird's-eye view of the Larynx from above.

GEH, the thyroïd cartilage embracing the ring of the cricoïd *ruXw*, and capable of turning on the axis *xx*, which passes through the lower horns *C*, Figs. 3, 4.

NF, *NF*, the arytenoïd cartilages.

TV, *TV*, the vocal ligaments.

NX, the right cricoarytenoïdeus lateralis, the left is removed.

Vkf, the left thyroarytenoïdeus, the right is removed.

N l, *N l*, cricoarytenoïdei postici.

The arytenoïdeus transversus is removed.

B, *B*, the ligaments uniting the arytenoïd and cricoïd.

FIG. 8 is part of Fig. 7 enlarged, to shew the direction of the muscular forces which act on the arytenoïd cartilage.

QNVS, the right arytenoïd.

TV, its vocal ligament.

BRS, the bundle of ligaments uniting it to the cricoïd.

OP, the projection of its axis of articulation.

hg the direction of the force of the thyroarytenoïdeus.

NX ——————————— cricoarytenoïdeus lateralis.

NW ——————————— cricoarytenoïdeus posticus.

NY ——————————— arytenoïdeus transversus.

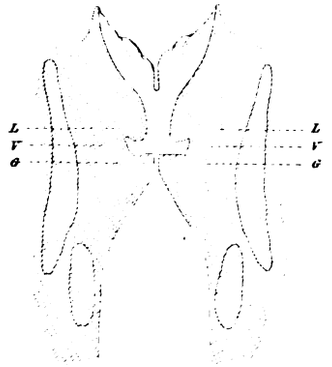
PLATE XXIII.

Consists of figures of apparatus and diagrams which are sufficiently explained in the text.

ERRATA.

Page 333, line 12 from bottom, for Figs. 7, 8, *read* Figs. 3, 7.

—— 338, — 6 from top, after "the point *S*," *insert* (Fig. 8).



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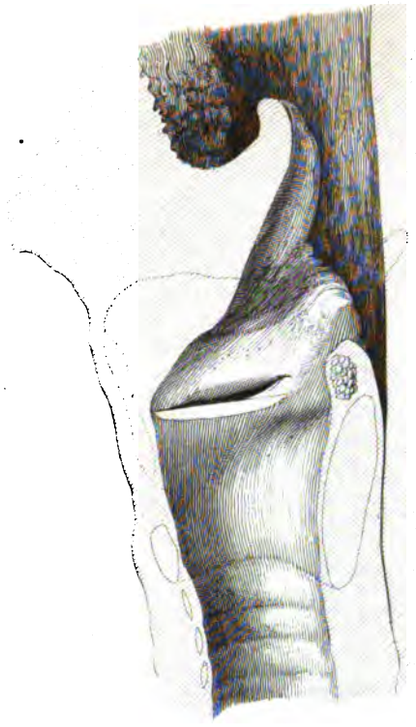

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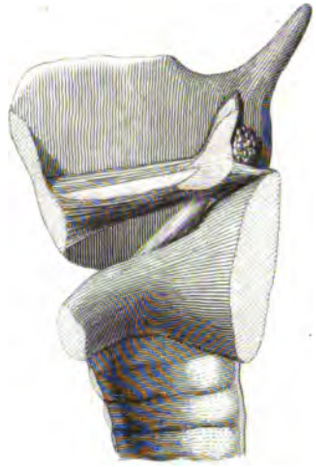

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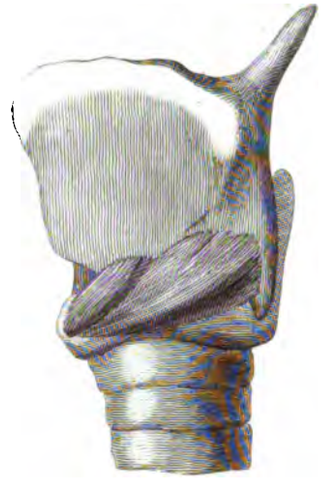

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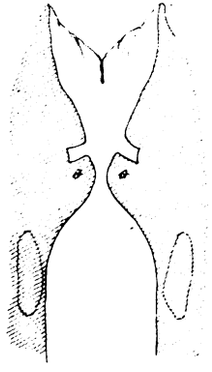

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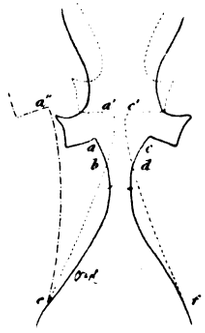

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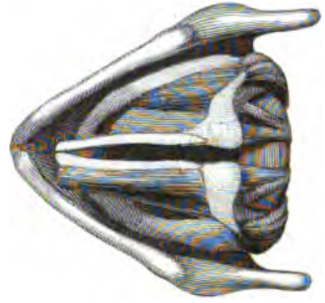

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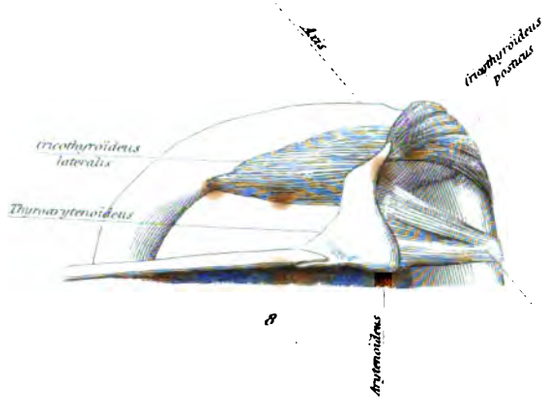

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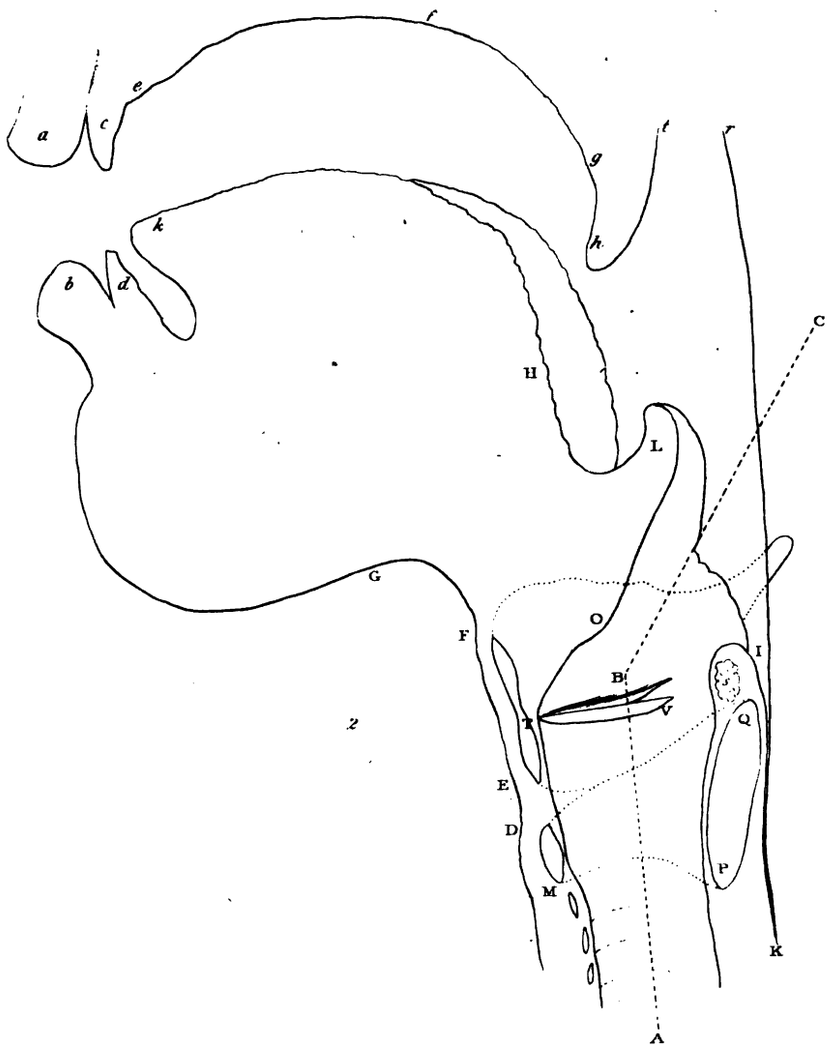

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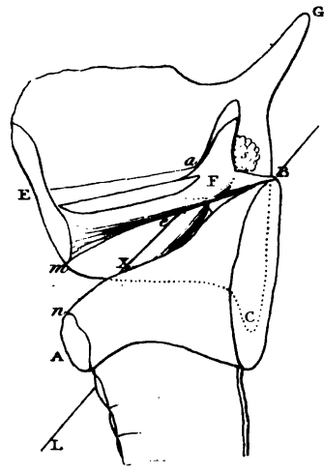

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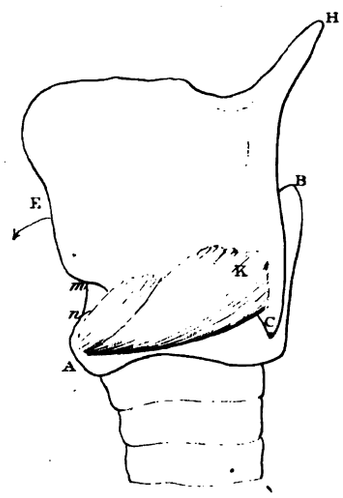

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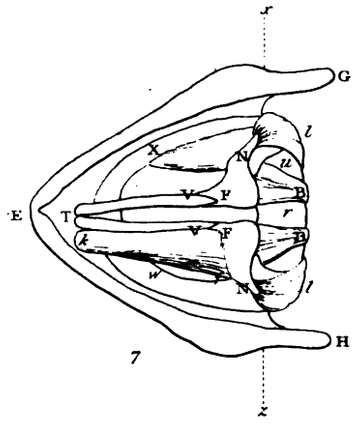

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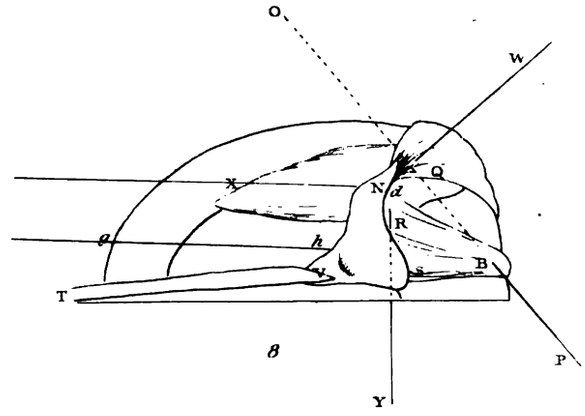

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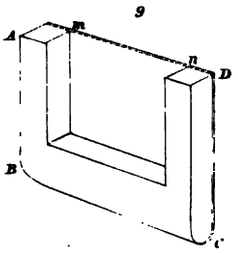
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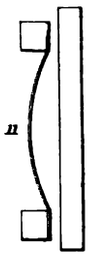
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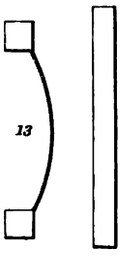
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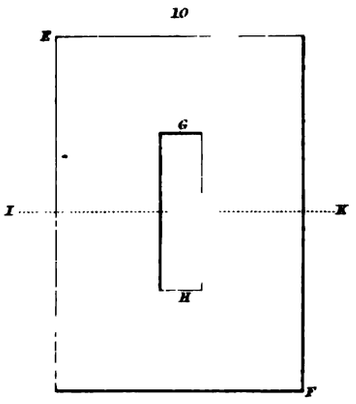
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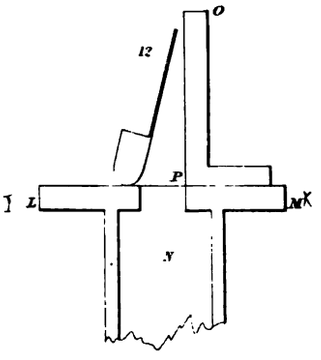
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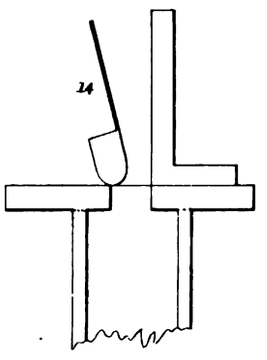
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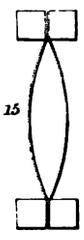
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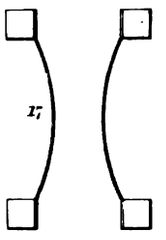
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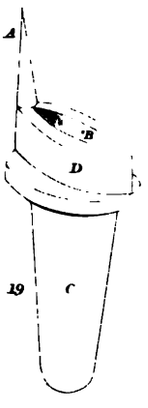
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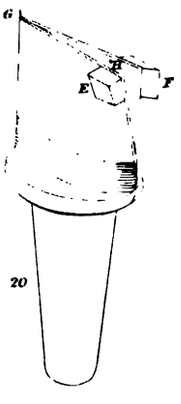
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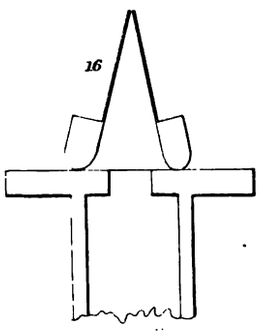
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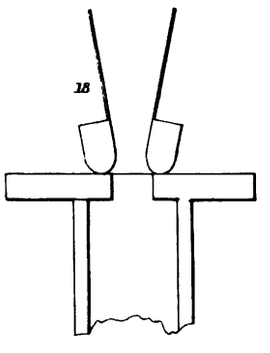
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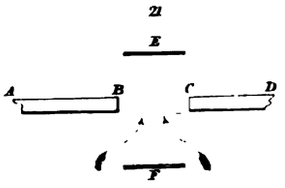
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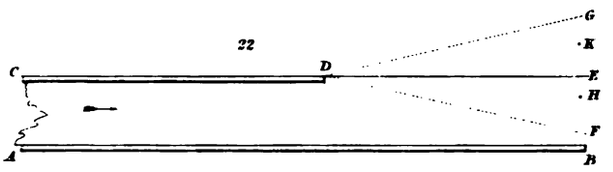
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XIII. *On the Inverse Method of Definite Integrals, with Physical Applications.*

By the Rev. R. MURPHY, B.A.

FELLOW OF CAIUS COLLEGE,

AND OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *March 5*, 1832.]

INTRODUCTION.

The mass of theory, on the subject of Definite Integrals, contributed by Fourier, Poisson, Cauchy, Gauss, and other modern analysts, is very considerable. Many of their properties, and applications have been pointed out, and the calculation of their numerical values facilitated. We may, therefore, regard the *direct calculus* of Definite Integrals as already formed, to a certain extent.

But as subjects are generally best understood, when examined on all sides, it would be advantageous, even in this point of view, to possess an *inverse* method of Definite Integrals, by which we may re-ascend from the known integral, to the unknown function under the sign of definite integration. At the same time objects of importance in the pure, and physical mathematics would be attained. Euler and Laplace have valued the

interpolated differential coefficients of fractional orders, for such functions as may be simply represented by Definite Integrals of a peculiar form. If we extend this principle to all functions of operation of the distributive kind*, (that is, such whose action on the whole, is the sum of the actions on the parts), and if we can represent a given algebraical function by a Definite Integral of the proper form, this view will be complete.

Again, the phænomena of the physical sciences generally result from an infinite number of the elementary actions of the particles forming the system under consideration. Such an inverse calculus would conduct us from the observed phænomenon, to the laws of the elementary actions.

The following researches have been conducted, with this object, chiefly, in view. The analysis consists of two parts, corresponding to two distinct classes of phænomena in nature. Namely, such as result from sensible, or developed powers, the action of which is generally insensible, at infinitely great distances; and such as are referred to latent, or *neutralized* powers, which only become sensible at indefinitely small distances from the component particles of the system, whence the actions emanate. Of the first part, which I now lay before the Society, the following is a brief abstract.

Adopting 0 and 1 throughout as the limits of integration, there is a character common to all the integrals of the functions commonly received in analysis; if the function be multiplied by t^r, that is, any positive power of the variable, the integral of the product, which is a function of x converges to 0, for infinitely great values of x. To revert from the function *outside*, to the

* Annales de Math. Tom. V. Servois' paper; Vid. also Tom. III. Francais' paper.

function *under* the definite integral, when the former is a rational function, we have only to multiply it by t^{-s} and taking the coefficient of $\frac{1}{x}$ in the product, divide it by t. This theorem, proved in Section (1), is there also extended to any number of variables; the remaining part of the Section illustrates the nature of the application of the theorem, and includes the mode of treating irrational functions.

The functions considered in the first Section are all continuous, the mode of treating *discontinuous* functions is shewn in the second; it was necessary to attend to this class, because the phænomena presented by nature are mostly of that kind. Thus the action of developed electricity follows two different laws, according as the point acted on, is within or without the surface of the body; the function which expresses this action generally, must therefore be discontinuous, the interruption taking place at the surface. In like manner heat is propagated simultaneously, in the earth, the sea, and the atmosphere, but according to different laws in each, and other instances of discontinuity are observable in every department of physics.

Now we shall obviate the difficulty thus presented by discontinuous functions, if we can obtain a formula, which without any alteration of form may continue to represent the function, under all circumstances. This object is attained by a simple application of the theory of Algebraic Equations. For when an Equation is resolved by the method given in my former paper*, the expression for the least root is in a form symmetrical with respect to all the roots. If for simplicity we only consider two

* Cambridge Philosophical Transactions, Vol. IV.

roots, one a constant, the other variable, commencing from nothing and increasing indefinitely; it is evident that the above formula, will represent the variable root, as long as it is less than the constant; but the instant the variable exceeds the constant root, the formula will cease to represent the former, and from thenceforward it will express the constant. This principle when fully developed, meets every case of discontinuity.

In the third Section, the preceding theory is applied, to the phænomena of developed electricity, from thence deducing the law of accumulation on the surface. The function which expresses the action of a closed conducting surface, charged in any conceivable manner with electricity is discontinuous, but the parts are not independent in consequence of the known law of force to each particle at different distances. By the principles of the first two Sections, we may, by observing the law of action of the electrised body, deduce the law of distribution on its surface. When for instance the action of a sphere, in any manner electrised is observed at distances greater than the radius to vary inversely as the $(n+2)$ power of the distance, the law of accumulation, which is expressed by a differential coefficient of the nth order, is of a remarkable kind. There will then be n nodal or transition lines on the surface, in which the electricity remains unresolved, and which divide the sphere into $n+1$ portions containing alternately the positive and negative electricities; in each portion there is a line of greatest accumulation (which becomes a point in the two extreme portions); the transition lines, and lines of greatest accumulation, divide the surface into belts, containing alternately in pairs, exactly equal quantities of the opposite fluids. And by the *superposition* of several systems of this kind, all the phænomena of developed electricity are produced.

When a sphere, already electrised, is subjected to the influence of an electrical point without it, the law of accumulation is simple; for the quantity of fluid which must be superposed to that already accumulated on each annulus, to make it every where equal to a certain constant, is always inversely proportional to the cube of the distance from the influencing point*.

As the application to the theory of developed electricity, offered a sufficient illustration to the principles of this part; at the same time that it conducted to new and remarkable results, it seemed unnecessary, considering the convenient limits of this paper, to insert other physical applications which I have made, particularly on the subjects of heat and magnetism.

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* The electrical action in the third Section, is measured by the tension of the fluid which *would* be produced in an infinitely thin rod, communicating with the electrical body, by the attraction or repulsion of the latter; it is what Mr Green, of Nottingham, in his ingenious Essay on this subject, has denominated the Potential Function.

SECTION I.

PRINCIPLES RELATIVE TO CONTINUOUS FUNCTIONS.

(1) *Preliminary Observations.*

1. FOR the purpose of uniformity, and simplicity, the integral limits which we take are 0 and 1*. If t be any variable with respect to which integrations are performed on any function, and the limits of t are the finite quantities a and b; then if we make $t=a+(b-a)t'$ the corresponding limits of the new variable, t' are 0 and 1. But if the limits of t are one finite and the other infinite, as a and $\pm\infty$, put then $t=a\pm\frac{t'}{1-t'}$; if they are both infinite as $-\infty$ and $+\infty$ make $t=\frac{1-2t'}{t'.(t'-1)}$; generally, therefore, by these substitutions, and by more convenient ones in particular cases, we may always make 0 and 1 to be the integral limits.

2. Let $f(t)$ be any function of t, and make

$$\phi(0)=\int f(t),\quad \phi(1)=\int f(t).t,\quad \phi(2)=\int f(t).t^2,\ \&c.$$

none of which definite integrals is supposed infinite. Then, x being any positive integer, we have $\phi(x)=\int f(t).t^x$. Put $1-t'$ for t (the limits of t' are also 0 and 1, changing the sign of the integral) and let the successive finite differences of $\phi(x)$ when x

* The limits uniformly adopted by Gauss. "*Methodus Nova Integr.*"

is made $=0$, be represented by $\Delta.\phi(0)$, $\Delta^2.\phi(0)$, &c. The equation when both its members are expanded becomes

$$\int f(t).t^x = \phi(0) + x.\Delta\phi(0) + \frac{x.(x-1)}{1.2}.\Delta^2\phi(0), \text{ \&c.}$$

$$= -\int_t f(t) + x\int_t f(t)\,t - \frac{x.(x-1)}{1.2}.\int_t f(t).t^2 + \text{\&c.}$$

and making x successively $=0$, 1, 2, 3, &c. we see that the corresponding terms in both series affected with their proper signs are exactly equal, the two series are therefore identical for *all values* of x. But the values of x from -1 to $-\infty$ are rejected as, *in general*, giving infinite integrals, for then if A_0 be the absolute term in $f(t)$ it is evident that $\int A_0 t^x$ will be infinite. In the particular case where $f(t)$ contains no positive power of t below t^m, we may then include the negative values of x, between 0 and $-m$. But in all cases when $\phi(0)$, $\phi(1)$, $\phi(2)$, &c. are finite, we may assign any value to x between -1 and 0, or between 0 and $+\infty$, and always have $\phi(x) = \int f(t).t^x$; we shall therefore in the following theory attribute to x all values included between -1 and $+\infty$ and suppose $\phi(x)$ to remain finite, during that interval.

3. Let us next consider the value of $\phi(x)$ when $x=\infty$, and $f(t)$ is any of the functions commonly received in analysis.

If $f(t)$ always remains finite between the extreme values $t=0$ and $t=1$, let the greatest value of $f(t)$ be represented by A and the least by A', then $\phi(x) = \int f(t).t^x$ is included between $\int A.t^x$ and $\int A'.t^x$, or between $\frac{A}{x+1}$ and $\frac{A'}{x+1}$; it is therefore $=0$ when x is infinite.

If $f(t)$, in the same interval, becomes infinite for a particular value of t (suppose when $t=a$), it is then of the form $\frac{P}{(t-a)^m}$, P being a finite function of t, of which the greatest and least values may be represented by p and p'; $\phi(x)$ then is between

$$p\int_t \frac{t^x}{(t-a)^m} \text{ and } p'\int_t \frac{t^x}{(t-a)^m}.$$

Now since

$$\int_t \frac{t^x}{(t-a)^m} = \frac{1}{x+1}\left\{\frac{1}{(1-a)^m} - \frac{1}{(-a)^m}\right\} + \frac{m}{(x+1).(x+2)}\cdot\left\{\frac{1}{(1-a)^{m+1}} - \frac{1}{(-a)^{m+1}}\right\} + \&c.$$

it is obvious that when $x=\infty$, $\phi(x)=0$ in this case also, with the exceptions which we are now about to examine.

When $a=0$, then $\phi(x)=\int_t \frac{P}{t^m}.t^x = \int_t P.t^{x-m}$, and putting for P its extreme values, $\phi(x)$ is evidently included between $\frac{p}{x-m+1}$ and $\frac{p'}{x-m+1}$, and is therefore nothing, when x is infinite; but when m is not less than unity, it is clear that $\phi(0)=\int_t \frac{P}{t^m}$ is infinite, the part of P not involving t, giving in this expression an infinite integral; this case is therefore inadmissible. (Art. 2.)

When $a=1$, then $\phi(x)$ is included between

$$p\int_t \frac{t^x}{(t-1)^m} \text{ and } p'\int_t \frac{t^x}{(t-1)^m};$$

if therefore m be not <1, $\phi(0)$ is included between

$$p\int_t \frac{1}{(t-1)^m} \text{ and } p'\int_t \frac{1}{(t-1)^m};$$

and is infinite, the absolute term in P giving as above, an infinite integral; rejecting therefore, this case, and supposing $m<1$, if we integrate by parts, we find

$$\int_t \frac{t^x}{(t-1)^m} = \frac{x}{x+(1-m)} \cdot \int_t \frac{t^{x-1}}{(t-1)^m};$$

this integral therefore converges to 0, as x increases to infinity.

Suppose next that $f(t)$ becomes infinite for several values of t, as $t=a$, $t=a'$, $t=a''$, &c.,

$$\text{then } f(t) \text{ is of the form } \frac{P}{(t-a)^m.(t-a')^{m'}.(t-a'')^{m''}.....},$$

which being resolved into simple fractions, the same reasoning as above shews that for all the admissible cases, we have $\phi(x)=0$ when $x=\infty$.

Lastly, let $f(t)$ be imaginary, so that $f(t)=P+Q\sqrt{-1}$, P and Q being real functions of t, then $\phi(x)=\int_t P.t^x+\sqrt{-1}\int_t Qt^x$, and by the above reasoning, both the latter functions vanish, and consequently $\phi(x)$ also $=0$, when $x=\infty$.

4. From this examination we see that the inverse problem "to recur from $\phi(x)$ to $f(t)$" consists of two parts*, first, when $\phi(x)$ converges to 0 as x increases to ∞, and is therefore essentially composed of negative powers of x; in this case which forms the subject of Part I. $f(t)$ is of the form of the usual functions received in analysis; secondly, when $\phi(x)$ does not consist of such powers, for instance when it is always zero, and then $f(t)$ belongs to a new and remarkable class of functions which will be treated

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* This is the division of Integrals into large classes, alluded to in p. 439, Vol. III. *Camb. Trans.*

of in Part II, and it has been stated in the Introduction, to what classes of natural phænomena, the respective parts correspond. It is obvious that the form of $\phi(x)$ when rational is $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+$&c. in Part I. We shall now proceed to the principles, by which we may revert from the given function $\phi(x)$ to the unknown function $f(t)$.

(2) *Inverse Method, for Rational Functions.*

5. ***When the known function*** $\phi(x)$ ***is rational, seek the coefficient of*** $\frac{1}{x}$ ***in*** $\phi(x).t^{-x}$; ***dividing it by*** t, ***the quotient will be the required function*** $f(t)$.

To prove this, suppose a to be any negative number; the coefficient of $\frac{1}{x}$ in $\phi(x).t^{-x}$, which we shall represent by T is also the coefficient of $\frac{1}{a}$ in $\phi(a).t^{-a}$. We shall get by actual integration from $t=0$ to $t=1$, the equation,

$$\ldots\ldots\ldots\int_t \frac{\phi(a)}{t^a}.t^{x-1}=\frac{\phi(a)}{x-a}\ldots\ldots(1),$$

and taking the coefficients of $\frac{1}{a}$ at both sides

$$\ldots\ldots\int T.t^{x-1}=\text{coefficient of } \frac{1}{a} \text{ in } \frac{\phi(a)}{x-a}\ldots\ldots(2)$$

Now, by supposition, $\phi(x)$ is of the form $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+$&c.

Consequently,

$$\frac{\phi(a)}{x-a}=\left\{\frac{A}{a}+\frac{B}{a^2}+\frac{C}{a^3}+\&c.\right\}.\left\{\frac{1}{x}+\frac{a}{x^2}+\frac{a^2}{x^3}+\&c.\right\}.$$

The coefficient therefore of

$$\frac{1}{a} \text{ in } \frac{\phi(a)}{x-a} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \&c.$$

$$= \phi(x).$$

Substituting this in equation (2) we have $\int_t \frac{T}{t} . t^x = \phi(x)$.....(3) which shews that $f(t) = \frac{T}{t}$ as announced above.

6. The following examples are intended to shew the manner, in which this theorem may be applied.

Ex. 1. Given $\int f(t) . t^x = \frac{1}{x+m}$, m being essentially positive, (vide Art. 2), to find $f(t)$.

We must find the coefficient of $\frac{1}{x}$ in $t^{-x} . \frac{1}{x+m}$ and divide it by t.

Now since $\frac{t^{-x}}{x+m}$

$$= \left\{1 - x \text{ h. l. } (t) + \frac{x^2}{1.2} . (\text{h. l. } t)^2 - \frac{x^3}{1.2.3} . (\text{h. l. } t)^3 + \&c.\right\}$$

$$\times \left\{\frac{1}{x} - \frac{m}{x^2} + \frac{m^2}{x^3} - \&c.\right\},$$

the coefficient of

$$\frac{1}{x} \text{ in } \frac{t^{-x}}{x+m} = 1 + m \text{ h. l. } (t) + \frac{m^2}{1.2} . (\text{h. l. } t)^2 + \frac{m^3}{1.2.3} . (\text{h. l. } t)^3 + \&c.$$

$$= t^m,$$

therefore the required function $f(t) = t^{m-1}$.

Ex. 2. Given $\int f(t) . t^x = \frac{1}{(x+m)^n}$ to find $f(t)$, (n being integer).

Since..... $\frac{t^{-x}}{(x+m)^n} = \{1 - x \text{ h. l. } (t) + \frac{x^2}{1.2} . (\text{h. l. } t)^2 - \&c.\}$

$$\times \{x^{-n} - n.mx^{-n-1} + \frac{n(n+1)}{1.2} . m^2 x^{-n-2} - \&c.\},$$

the coefficient of $\frac{1}{x} = \frac{\left(\text{h. l. } \frac{1}{t}\right)^{n-1}}{1.2.3...(n-1)} . \{1 + m \text{ h. l. } (t) + \frac{m^2 (\text{h. l. } t)^2}{1.2} + \&c.\}$

$$= \frac{t^m . \left(\text{h. l. } \frac{1}{t}\right)^{n-1}}{1.2.3....(n-1)},$$

and therefore in this case $f(t) = \frac{t^{m-1} . \left(\text{h. l. } \frac{1}{t}\right)^{n-1}}{1.2.3 ....(n-1)}$.

Ex. 3. Given $\int f(t) . t^x = \text{h. l. } \left(1 + \frac{a}{x+m}\right)$, m being essentially positive, and a either positive or between 0 and $-m$.

In this case we have

$$t^{-x} . \text{h. l. } \left(1 + \frac{a}{x+m}\right) = t^m . \{1 - (x+m) \text{ h. l. } (t) + \frac{(x+m)^2}{2} . (\text{h. l. } t)^2 - \&c.\}$$

$$\times \left\{\frac{a}{x+m} - \frac{1}{2} . \frac{a^2}{(x+m)^2} + \&c.\right\},$$

and if we select the coefficient of $\frac{1}{x}$ from the products of the corresponding terms in both series (which are the only products that contain $\frac{1}{x}$) we get, the coefficient of $\frac{1}{x}$ in

$$t^{-x} \text{h. l. } \left(1 + \frac{a}{x+m}\right) = t^m . \{a + \frac{a^2}{1.2} . \text{h. l. } (t) + \frac{a^3}{1.2.3} . (\text{h. l. } t)^2 \ \&c.\}$$

$$= t^m . \frac{t^a - 1}{\text{h. l. } (t)},$$

and therefore $f(t) = t^{m-1} . \frac{t^a - 1}{\text{h. l. } (t)}$.

(3) *Means of facilitating the Calculus of* $f(t)$.

7. By this method we may always revert from $\phi(x)$ to $f(t)$; but to facilitate the process, it will be found convenient to reduce $\phi(x)$ to its simplest form, previous to the application of the general theorem.

Thus suppose $\phi(x)$ expanded according to the descending powers of $x+m$ (m being any positive quantity) i. e. if

$$\phi(x) = \frac{A_0}{x+m} + \frac{A_1}{(x+m)^2} + \frac{A_2}{(x+m)^3} + \&c.$$

then assuming the result of the second Example above, we get,

$$f(t) = t^{m-1}\left\{A_0 + A_1 \text{h.l.}\left(\frac{1}{t}\right) + \frac{A_2 \cdot \left(\text{h.l.}\frac{1}{t}\right)^2}{1.2} + \frac{A_3 \cdot \left(\text{h.l.}\frac{1}{t}\right)^3}{1.2.3} + \&c.\right\}.$$

8. When $\phi(x)$ is a rational fraction, the highest power of x in the denominator being greater than that in the numerator, and the denominator not vanishing in the interval from $x=0$ to ∞, then if we decompose $\phi(x)$ into its simple fractions, the results of the first and second Examples above will give us the value of $f(t)$.

Ex. 4. Given

$$\phi(x) = \frac{x.(x-1).(x-2).....(x-n+1)}{(x+1)(x+2)(x+3).....(x+n+1)};$$

to find $f(t)$. Put

$$\phi(x) = \frac{N_1}{x+1} + \frac{N_2}{x+2} + \ldots \frac{N_{n+1}}{x+n+1},$$

which compared with the above, gives

$$x.(x-1).(x-2)\ldots(x-n+1) = N_1(x+2)(x+3)\ldots(x+n+1)$$
$$+ N_2(x+1)(x+3)\ldots(x+n+1) + \&c.$$

and to determine N_1, N_2 &c. suppose, successively, $x=-1, -2, -3$, &c. thus we obtain

$$N_1=(-1)^n;\ N_2=-(-1)^n.\frac{n}{1}.\frac{n+1}{1};\ N_3=(-1)^n.\frac{n.(n-1)}{1.2}.\frac{(n+1).(n+2)}{1.2}+\&c.$$

and therefore

$$\phi(x)=(-1)^n.\left\{\frac{1}{x+1}-\frac{n}{1}.\frac{n+1}{1}.\frac{1}{x+2}+\frac{n.(n-1)}{1.2}.\frac{(n+1).(n+2)}{1.2}.\frac{1}{x+3}-\&c.\right\},$$

and by Ex. 2 we have therefore

$$f(t)=(-1)^n\{1-\frac{n}{1}.\frac{n+1}{1}.t+\frac{n.(n-1)}{1.2}.\frac{(n+1).(n+2)}{1.2}.t^2-\&c.\},$$

an elegant result, to which we shall hereafter have occasion to refer.

Ex. 5. Given $\phi(x)=\dfrac{1}{(x+h).(x+2h).....(x+nh)}$.

Applying the same method we get in this case

$$f(t)=\frac{t^{h-1}.(1-t^h)^{n-1}}{1.2.3...(n-1).h^{n-1}}.$$

9. Let us next consider logarithmic functions, beginning with h. l. $\dfrac{P}{Q}=\phi(x)$, supposing that neither P nor Q vanishes from $x=0$ to ∞, and that the highest powers in the rational functions P and Q may be equal, and be multiplied by equal coefficients; which conditions are evidently necessary to make $\phi(x)$ remain finite for all positive values of x, and to vanish when $x=\infty$.

To find $f(t)$, let P and Q be resolved into their simple fractions, so that

$$P=C.(x+a_1)(x+a_2).....(x+a_n),$$
$$\text{and } Q=C.(x+b_1).(x+b_2).....(x+b_n);$$

therefore $\text{h.l.}\frac{P}{Q} = \text{h.l.}\left(1+\frac{a_1-b_1}{x+b_1}\right)+\text{h.l.}\left(1+\frac{a_2-b_2}{x+b_2}\right)+\&\text{c.} = \phi(x)$.

Hence by Ex. 3,

$$f(t) = \frac{1}{t\,\text{h.l.}(t)}.\{(t^{a_1}+t^{a_2}+\ldots t^{a_n})-(t^{b_1}+t^{b_2}+\ldots t^{b_n})\}.$$

For other instances of Logarithmic functions, we refer to Note (A) at the end of this paper.

10. Another form of $\phi(x)$ to be considered, as frequently occurring, is that of fractions of which the numerator and denominator are composed of a variable number of factors; the relation between $\phi(x)$ and $\phi(x+1)$ will enable us to transform such a function into a series of simple fractions; we may then recur from $\phi(x)$ to $f(t)$ as before.

Ex. 6. Given $\phi(x) = \frac{2.4.6.....2x}{3.5.7.....(2x+1)}$, to find $f(t)$.

By the nature of this function we have

$$(2x+2).\phi(x) = (2x+3).\phi(x+1).$$

Substitute for $\phi(x)$ the series

$$\frac{A}{x+n}+\frac{B}{x+n+1}+\frac{C}{x+n+2}\ \&\text{c.};$$

the equation becomes by actual division,

$$\frac{2A.(1-n)}{x+n}+\frac{2B.(1-n-1)}{x+n+1}+\frac{2C.(1-n-2)}{x+n+2}\ \&\text{c.}$$

$$=\frac{2A.\left(\frac{3}{2}-n-1\right)}{x+n+1}+\frac{2B.\left(\frac{3}{2}-n-2\right)}{x+n+2}\ \&\text{c.}$$

and putting $n=1$ to make both series comparable, term by term, we get

$$\frac{2B}{x+2}+\frac{4C}{x+3}+\frac{6D}{x+4}, \text{ \&c.} = \frac{A}{x+2}+\frac{3B}{x+3}+\frac{5C}{x+4}, \text{ \&c.}$$

$$\text{which gives } B=A.\frac{1}{2},\quad C=B.\frac{3}{4},\quad D=C.\frac{5}{6} \text{ \&c.}$$

$$\text{so that } \phi(x)=A\left\{\frac{1}{x+1}+\frac{1}{2}.\frac{1}{x+2}+\frac{1.3}{2.4}.\frac{1}{x+3}+\text{\&c.}\right\};$$

$$\text{and } \therefore f(t)=A\left\{1+\frac{1}{2}.t+\frac{1.3}{2.4}.t^2+\text{\&c.}\right\}=\frac{A}{(1-t)^{\frac{1}{2}}};$$

the constant A is found by giving x a particular value as 1,

$$A\int_t \frac{t}{\sqrt{1-t}}=\frac{2}{3},$$

which gives $A=\frac{1}{2}$.

Ex. 7. Given

$$\phi(x)=\frac{a.(a+a).(a+2a).....\{a+(x-1).a\}}{(a+\beta).(a+\beta+a).(a+\beta+2a).....\{a+\beta+(x-1)a\}} \text{ to find } f(t).$$

Following the same steps, we find here

$$n=\frac{a}{a},\quad B=-A\left(\frac{\beta}{a}-1\right),\quad C=-\frac{B}{2}.\left(\frac{\beta}{a}-2\right) \text{ \&c.}$$

$$\text{whence } f(t)=A.t^{\frac{a}{a}-1}.(1-t)^{\frac{\beta}{a}-1}.$$

A similar method is to be used, when the relation between $\phi(x)$ and $\phi(x+1)$ is of a more complicated nature, or when that relation refers to more than two such functions.

11. The denominators of the simple fractions in the two preceding articles have been real; such quadratic factors as would

introduce imaginary quantities, when decomposed into simple factors, may more conveniently be left in the quadratic form, as in the following example.

Ex. 8. Given $\phi(x) = \dfrac{Ax+B}{(x+a)^2+\beta^2}$.

It is easily seen that

$$\phi(x) = A\left\{\frac{1}{x+a} - \frac{\beta^2}{(x+a)^3} + \frac{\beta^4}{(x+a)^5} - \&c.\right\}$$

$$+(B-Aa)\left\{\frac{1}{(x+a)^2} - \frac{\beta^2}{(x+a)^4} + \frac{\beta^4}{(x+a)^6} - \&c.\right\}$$

and therefore by Ex. 2.

$$f(t) = At^{a-1}.\left\{1 - \frac{\beta^2\left(\text{h.l.}\frac{1}{t}\right)^2}{1.2} + \frac{\beta^4\,\text{h.l.}\left(\frac{1}{t}\right)^4}{1.2.3.4} - \&c.\right.$$

$$\left. + \frac{(B-Aa)}{\beta}t^{a-1}\left\{\frac{\beta\,\text{h.l.}\frac{1}{t}}{1} - \frac{\beta^3\left(\text{h.l.}\frac{1}{t}\right)^3}{1.2.3} + \&c.\right\}\right.$$

$$= t^{a-1}\left\{A\cos\left(\beta\,\text{h.l.}\frac{1}{t}\right) + \frac{B-Aa}{\beta}.\sin\left(\beta\,\text{h.l.}\frac{1}{t}\right)\right\}$$

put $\mu^2 = A^2 + \left(\dfrac{B-Aa}{\beta}\right)^2$ and $\gamma = \tan^{-1}.\dfrac{A\beta}{B-Aa}$.

$$\text{Hence } f(t) = t^{a-1}.\mu\sin\left(\gamma+\beta\,\text{h.l.}\frac{1}{t}\right);$$

which result if differentiated n times with respect to β^2 as independent variable, will solve the case of n equal quadratic factors.

(4) *Inverse Method for Irrational Functions, and Functions of several Variables.*

12. None of the forms of $\phi(x)$ hitherto taken, have involved irrational quantities, but a simple modification of the preceding results, will make them include surds. By Ex. 2,

$$\text{if } \phi(x) = \frac{1}{(x+m)^n}, \text{ then } f(t) = \frac{t^{m-1}\left(\text{h.l.}\frac{1}{t}\right)^{n-1}}{1.2.3\ldots(n-1)},$$

and to make this apply also to the fractional values of n, we have only to put the denominator under its more general form $\int_t \left(\text{h.l.}\frac{1}{t}\right)^{n-1}$.

To prove this put $t^{x+m}=\tau$, the limits of τ are the same as those of t,

$$\text{and } \int_t t^{m-1}.\left(\text{h.l.}\frac{1}{t}\right)^{n-1}.t^x = \frac{1}{(x+m)^n}\int_\tau \left(\text{h.l.}\frac{1}{\tau}\right)^{n-1},$$

$$\text{but } \int_t \left(\text{h.l.}\frac{1}{t}\right)^{n-1} = \int_\tau \left(\text{h.l.}\frac{1}{\tau}\right)^{n-1},$$

from whence it is obvious that $f(t) = \dfrac{t^{m-1}\left(\text{h.l.}\frac{1}{t}\right)^{n-1}}{\int_t \left(\text{h.l.}\frac{1}{t}\right)^{n-1}}$.

*Ex. 9. Given $\phi(x) = \dfrac{e^{-\frac{a^2}{x+m}}}{\sqrt{x+m}}$ to find $f(t)$.

If we expand $\phi(x)$, the form of the general term, abstracting from the sign, is $\dfrac{a^{2n}}{1.2.3\ldots n.(x+m)^{n+\frac{1}{2}}}$.

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* Laplace, *Theor. des Prob.* p. 97. Ed. 1820.

To make the limits 0 and 1 in Laplace's formula, put $\epsilon^{-a^2}=t$; and change the sign of the integral.

The corresponding term in $f(t)$, is

$$\frac{a^{2n}}{1.2.3...n} \cdot \frac{t^{n-1} \cdot \left(\text{h.l.} \frac{1}{t}\right)^{n-\frac{1}{2}}}{\int_t \left(\text{h.l.} \frac{1}{t}\right)^{n-\frac{1}{2}}}$$

But $\int_t \left(\text{h.l.} \frac{1}{t}\right)^{n-\frac{1}{2}} = \frac{1.3.5...(2n-1)}{2^n} \cdot \int_t (\text{h.l.}\, t)^{-\frac{1}{2}} = \sqrt{\pi} \cdot \frac{1.3.5...(2n-1)}{2^n}$,

thus the general term of $f(t)$, abstracting from the sign, becomes

$$\frac{t^{n-1}}{\sqrt{\pi\, \text{h.l.} \left(\frac{1}{t}\right)}} \cdot \frac{(2a)^{2n} \cdot \left(\text{h.l.} \frac{1}{t}\right)^n}{1.2.3...2n};$$

$$\text{from whence } f(t) = \frac{t^{n-1} \cos \cdot \left\{2a \sqrt{\text{h.l.} \left(\frac{1}{t}\right)}\right\}}{\sqrt{\pi\, \text{h.l.} \left(\frac{1}{t}\right)}}.$$

13. In applying the above principles for recurring from $\phi(x)$ to $f(t)$ in the equation $\int_t f(t).t^x = \phi(x)$, we should attend to the extent of the values to which x is limited.

When $\phi(x)$ remains finite from $x=0$ to $x=\infty$ we may assign to x any value from $x=-1$ to $x=\infty$. (Art. 2).

When $\phi(x)$ remains finite from $x=h$ to $x=\infty$, i. e. if h be the greatest real root of the equation $\frac{1}{\phi(x)}=0$, we may assign to x any value from $x=h-1$ to $x=\infty$. For the equation may be put under the form $\int_t f(t).t^h.t^{x-h} = \phi(x)$; put $x-h=x'$, and $f(t).t^h = F(t)$, $\therefore \int_t F(t).t^{x'} = \phi(x'+h)$, which remaining finite from $x'=0$ to $x'=\infty$ we may assign to x' all values from $x'=-1$ to $x'=+\infty$, i. e. we may put for x any number between $h-1$ and $+\infty$.

When $\phi(x)$ remains finite for all real values of x, we may in the applications give x any value from $-\infty$ to $+\infty$, and may then use formula of the form

$$\int_t f(t).(t^x \pm t^{-x}) = \phi(x) \pm \phi(-x).$$

If $\phi(x)$ remains finite for all values of x from $x=-b$ to $x=b$, then putting $t=t'^n$ and $x=\frac{x'}{n}$ the equation becomes

$$n \int_t f(t'^n).t'^{n-1}.t'^{x'} = \phi\left(\frac{x'}{n}\right),$$

which integral remains finite from $x'=-nb$ to $x'=nb$, and if we increase n indefinitely these limits tend to $-\infty$ and $+\infty$, and therefore we may use the formula

$$\int_t f(t'^{n-1}).t'^{n-1}(.t'^{x'} \pm t'^{-x'}) = \phi\left(\frac{x'}{n}\right) \pm \phi\left(\frac{-x'}{n}\right),$$

for all values of x' between $-nb$ and $+nb$, or which is the same, we may use

$$\int_t f(t).(t^x \pm t^{-x}) = \phi(x) \pm \phi(-x),$$

for such values of x as lie between $-b$ and $+b$.

* **Ex. 10.** Given

$$\int_t f(t).\{t^x - t^{-x}\} = \frac{\pi}{2}\tan\left(\frac{\pi}{2}x\right), \text{ to find } f(t).$$

Since $\frac{\pi}{2}\tan\left(\frac{\pi}{2}x\right) = \frac{1}{1-x} - \frac{1}{1+x} + \frac{1}{3-x} - \frac{1}{3+x}$ &c.

$$= \phi(x) - \phi(-x) \ldots \text{putting } \phi(x) = -\left\{\frac{1}{1+x} + \frac{1}{3+x} + \&c.\right\},$$

and it remains finite from $x=-1$ to $+1$, then deducing $f(t)$ from $\phi(x)$ by the usual method, we have

$$f(t) = -\{1 + t^2 + t^4 + \&c.\} = \frac{1}{t^2-1}.$$

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* Vide Cauchy's paper, *Annales de Math.* Tom. XVII. Formula 115.

When the highest negative power which enters $f(t)$ is t^{-h}, then $\int_t f(t)\,t^x$ (with the limits of t, 0 and 1), is only finite when $x>(h-1)$, but if x be $<h-1$, the parts of the integral which then become infinite retain their former finite values by making the limits of t to be ∞ and 1; and all the principles established in this section will apply to them with the new limits.

14. Let f now denote a function of any number of variables t, t', t'' &c. and ϕ, a function of other variables x, x', x'' &c. under the same restrictions, with respect to each of the variables as before, and let all the limits of integration be 0 and 1; then if there is given the equation

$$\int_t \int_{t'} \int_{t''} \ldots\ldots f.t^x.t'^{x'}.t''^{x''} \ldots\ldots = \phi,$$

we may revert from ϕ to f, by the following theorem:

"Take the coefficient of

$$\frac{1}{xx'x''\ \&c.} \text{ in } \phi.t^{-x}.t'^{-x'}.t''^{-x''} \ldots\ldots$$

and divide it by $tt't''$...&c.; the quotient will be f."

For let $t.\phi'$ be the coefficient of $\frac{1}{x}$ in $\phi.t^{-x}$,

$$t'.\phi'' \ldots\ldots\ldots\ldots\ldots\ldots \frac{1}{x'} \text{ in } \phi'.t'^{-x'} \text{ or of } \frac{1}{xx'} \text{ in } \phi t^{-x} t'^{-x'},$$

and so on, then (Art. 5) $\int_t \phi'.t^x = \phi$,

$$\int_{t'} \phi''.t'^{x'} = \phi', \quad \therefore \int_t \int_{t'} \phi''.t^x t'^{x'} = \phi;$$

and the same reasoning evidently holds for any number of variables.

It will not be necessary to annex any examples, to elucidate the applications of this theorem, as they are similar to the examples for functions of only one variable. (Vid. Note B.)

SECTION II.

PRINCIPLES RELATIVE TO DISCONTINUOUS FUNCTIONS.

(1). *Method of representing Discontinuous Functions of only one Break.*

15. To bestow on the principles of this analysis, all the generality, required by physical problems, we must extend it to such cases of definite integrals, as undergo a total change in their values, under given circumstances. The attraction of a spherical shell for instance, on any point is a definite integral. But when the thickness is indefinitely small and uniform, and each particle attracts by a force varying inversely as the square of the distance, this definite integral is 0 when the attracted particle is within, and a finite quantity when without the shell. The law of attraction is therefore interrupted when the attracted point arrives at the surface of the shell. When the shell possesses a finite thickness there are two *breaks* in the attraction, or the definite integral by which it is expressed. The attractions of spheroids of every form are similarly discontinuous, when the force follows such a law that it becomes insensible at great distances. So also in heated bodies; the law of expansion is different for the same body, in the different states of solid, fluid, and gaseous. The electrical phænomena present several similar instances. To obviate the difficulty thus offered, we must seek *an analytical expression, which without altering its form, may continue to represent the true value of the definite integral, or discontinuous function, under all circumstances.*

16. To find a formula, which shall always represent the *least* of two quantities α and β, abstracting from the sign.

It has been shewn, in the paper on the Resolution of Equations*, that if $\phi(x)$ be any function of x involving only positive and integer powers of x, the coefficient of $\frac{1}{x}$ in $-\text{h.l.}\frac{\phi(x)}{x}$ is the same root of the equation $\phi(x)=0$ as that given by Lagrange's theorem, that is, the least. Hence it follows that if we seek the coefficient of $\frac{1}{x}$ in $\text{h.l.}\frac{(x+\alpha).(x+\beta)}{x}$ or in $\text{h.l.}\left(1+\frac{\alpha\beta+x^2}{x(\alpha+\beta)}\right)$, it will represent the least of the quantities α or β. Expanding, therefore, this logarithm, and taking the coefficient of $\frac{1}{x}$ in each term, we get the equation,

$$\text{The least of } \begin{cases}\alpha\\ \beta\end{cases} = \frac{\alpha\beta}{\alpha+\beta} + \frac{1.1}{2.4}\cdot\frac{(\alpha\beta)^2}{\left(\frac{\alpha+\beta}{2}\right)^3} + \frac{1.1.3}{2.4.6}\cdot\frac{(\alpha\beta)^3}{\left(\frac{\alpha+\beta}{2}\right)^5} + \&c.$$

17. Put $\frac{1}{\alpha}$ for α and $\frac{1}{\beta}$ for β; we find thus:

$$\text{The least of } \begin{cases}\frac{1}{\alpha}\\ \frac{1}{\beta}\end{cases} = \frac{1}{\alpha+\beta} + \frac{1.1}{2.4}\cdot\frac{\alpha\beta}{\left(\frac{\alpha+\beta}{2}\right)^3} + \frac{1.1.3}{2.4.6}\cdot\frac{(\alpha\beta)^2}{\left(\frac{\alpha+\beta}{2}\right)^5} + \&c.$$

representing this series by S, it follows that

$$\frac{d^{n-1}.S}{(-1)^{n-1}.1.2.....(n-1).d\alpha^{n-1}} = \frac{1}{\alpha^n} \text{.....when } \alpha>\beta,$$

$$\text{and } =0 \text{ when } \alpha<\beta,$$

that is, we have a formula which represents 0 or $\frac{1}{\alpha^n}$ according as α is $<$ or $>$ β.

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* Camb. Trans. Vol. IV. Part I.

Thus if a represent the distance of any point, from the centre of a spherical shell, the formula which represents the attraction both within and without $\left(\text{i. e. } 0 \text{ and } \frac{1}{a^2}\right)$ is $-\frac{dS}{da}$.

18. To find a formula which shall represent $f(a)$ when $a<\beta$, and $f(\beta)$ when a is greater than β; $f(a)$ being supposed to consist of positive powers of a.

It appears from the paper above referred to, that the object will be answered by taking the coefficient of $\frac{1}{x}$ in $-f'(x)\,\text{h.l.}\frac{\phi(x)}{x}$; where $f'(x)$ is the derived function of $f(x)$ and $\phi(x)=(x-a).(x-\beta)$.

Thus, if $f(a)=a^m$, m being positive, we must take the coefficient of $\frac{1}{x^m}$ in $-m\,\text{h.l.}\left(1-\frac{a\beta+x^2}{x(a+\beta)}\right)$; that is, the term independent of x in

$$m\left\{\frac{1}{m}\left(\frac{a\beta+x^2}{a+\beta}\right)^m+\frac{1}{x.(m+1)}\left(\frac{a\beta+x^2}{a+\beta}\right)^{m+1}+\frac{1}{x^2.(m+2)}.\left(\frac{a\beta+x^2}{a+\beta}\right)^{m+2}+\&c.\right\},$$

which evidently is

$$\left\{\frac{(a\beta)^m}{(a+\beta)^m}+\frac{(m)}{1}.\frac{(a\beta)^{m+1}}{(a+\beta)^{m+2}}+\frac{(m+3).(m)}{1.2}.\frac{(a\beta)^{m+2}}{(a+\beta)^{m+4}}+\&c.\right\},$$

the general term of which is

$$\frac{m.(m+n+1).(m+n+2)\ldots(m+2n-1)}{1.2.3\ldots n}.\frac{(a\beta)^{m+n}}{(a+\beta)^{m+2n}}.$$

COR. Putting $\frac{1}{a}$ for a, and $\frac{1}{\beta}$ for β; the following formula will represent $\frac{1}{a^m}$ when $a>\beta$, and $\frac{1}{\beta^m}$ when $\beta>a$, viz.

$$\left\{\frac{1}{(a+\beta)^m}+\frac{m}{1}.\frac{a\beta}{(a+\beta)^{m+2}}+\frac{m.(m+3)}{1.2}.\frac{(a\beta)^2}{(a+\beta)^{m+4}}\ \&c.\right\}.$$

19. To find a formula, of which the value is $\frac{1}{\alpha-h\beta}$, when $\alpha>\beta$, and $\frac{1}{\beta-h\alpha}$ when $\beta>\alpha$; h being any quantity less than unity.

The order, in point of magnitude, of the quantities $\frac{1}{\alpha-h\beta}$ and $\frac{1}{\beta-h\alpha}$, is the same as the order of $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. For suppose $\alpha<\beta$ then $\alpha+h\alpha<\beta+h\beta$; and, subtracting $h(\alpha+\beta)$ from both sides, we have $\alpha-h\beta<\beta-h\alpha$, attributing to h any value between 0 and $\frac{\alpha}{\beta}$; again since $h<\beta$ we also have $\beta h+\alpha h<\beta+\alpha$; subtract $\alpha h+\alpha$ from both sides, $\therefore \beta h-\alpha<\beta-\alpha h$, attributing to h now any value between $\frac{\alpha}{\beta}$ and 1; whence it is evident that for any value of h from 0 to 1, we always have $\alpha-\beta h$ (abstracting from its sign)$<\beta-\alpha h$, when $\alpha<\beta$: and therefore if $\frac{1}{\alpha}$ be $>$, $=$, or $<\frac{1}{\beta}$ accordingly shall $\frac{1}{\alpha-\beta h}$ be $>$, $=$, or $<\frac{1}{\beta-\alpha h}$.

Now by Art. 17, we have

$$\text{the least of } \left\{\begin{matrix}\frac{1}{\alpha-\beta h}\\ \frac{1}{\beta-\alpha h}\end{matrix}\right. = \frac{1}{(\alpha+\beta)(1-h)}$$

$$+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2^2\{\alpha\beta.(1+h)^2-h.(\alpha+\beta)^2\}}{(\alpha+\beta)^3.(1-h)^3}+\frac{1.3}{2.4}\cdot\frac{1}{3}\cdot\frac{2^4\{\alpha\beta.(1+h)^2-h.(\alpha+\beta)^2\}^2}{(\alpha+\beta)^5.(1-h)^5}+\&c.\Bigg\},$$

which is obtained by putting $\alpha-\beta h$ for α, and $\beta-\alpha h$ for β in the article referred to.

This series may be easily arranged in the more convenient form,

$$B_0.\frac{1}{\alpha+\beta}+\frac{1}{2}.B_1.\frac{2^2\alpha\beta}{(\alpha+\beta)^3}+\frac{1.3}{2.4}.B_2.\frac{2^4(\alpha\beta)^2}{(\alpha+\beta)^5}+\&c.$$

where $B_n=(1+h)^{2n}$

$$\times\left\{\frac{1}{(1-h)^{2n+1}}.\frac{1}{n+1}-\frac{(2n+1).2^2h}{2(1-h)^{2n+3}}.\frac{1}{n+2}+\frac{(2n+1).(2n+3).2^4h^2}{2.4.(1-h)^{2n+5}}.\frac{1}{n+3}\ \&c.\right\}$$

$$=\int_t\frac{(1+h)^{2n}.t^n}{\{(1-h)^2+2^2ht\}^{n+\frac{1}{2}}}\text{ from }t=0\text{ to }t=1.$$

20. To find a formula, which shall represent $\frac{\beta^n}{\alpha^{n+1}}$, when $\alpha>\beta$, and $\frac{\alpha^n}{\beta^{n+1}}$, when $\beta>\alpha$.

Suppose that H_n is the coefficient of h^n in B_n, in the last Article, it follows that

$$H_0.\frac{1}{\alpha+\beta}+\frac{1}{2}.H_1.\frac{2^2\alpha\beta}{(\alpha+\beta)^3}+\frac{1.3}{2.4}.H_2.\frac{2^4(\alpha\beta)^2}{(\alpha+\beta)^5}+\&c.$$

is the coefficient of h^n in the least of $\begin{cases}\frac{1}{\alpha-h\beta}\\ \frac{1}{\beta-h\alpha}\end{cases}$, i. e. it represents $\frac{\beta^n}{\alpha^{n+1}}$ or $\frac{\alpha^n}{\beta^{n+1}}$, according as $\alpha>$ or $<\beta$.

Integrating by parts, and observing that $B_0=$ the least of $\begin{cases}\frac{1}{1}\\ \frac{1}{-h}\end{cases}=1$; we get

$$B_n=-\left\{\frac{1}{2n-1}.\frac{1+h}{2h}+\frac{2n.(1+h)^3}{(2n-1).(2n-3).(2^3h^2)}\right.$$
$$\left.+\frac{2n.(2n-2)}{(2n-1).(2n-3).(2n-5)}.\frac{(1+h)^5}{2^5h^3}+\&c.\right\}$$
$$+\frac{2.4.6\ldots2n}{1.3.5\ldots(2n-1)}.\frac{(1+h)^{2n}}{2^{2n}h^n}.$$

Comparing this expression with the series for B_n in the last Article, with which it ought to be identical; we may simplify, by expanding each term in that series according to the powers of h, and reject all powers higher than h^n, since they must mutually destroy each other; or we may expand each term in the latter expression, and reject the negative powers of h for the same reason.

Adopting the latter process, it is obvious that the part of B_n between the brackets, is of the form

$$-\left\{\begin{matrix} A_0 + A_1 h + A_2 h^2 + \&c. \\ + \frac{A_0}{h} + \frac{A_1}{h^2} + \frac{A_2}{h^3} + \&c. \end{matrix}\right\},$$

and the part without $= \frac{2.4.6\ldots 2n}{1.3.5\ldots(2n-1)} \cdot \frac{1}{2^{2n}}$

$$\left\{\begin{matrix} \frac{2n.(2n-1)\ldots(n+1)}{1.2\ldots n} + \frac{2n.(2n-1)\ldots(n+2)}{1.2\ldots(n-1)}.h + \frac{2n.(2n-1)\ldots(n+3)}{1.2\ldots(n-2)}\ h^2\ \&c. \\ + \frac{2n.(2n-1)\ldots(n+2)}{1.2\ldots(n-1)} \cdot \frac{1}{h} + \frac{2n.(2n-1)\ldots(n+3)}{1.2\ldots(n-2)} \cdot \frac{1}{h^2}\ \&c. \end{matrix}\right\}:$$

but since the negative powers of h must disappear, this consideration gives

$$A_0 = \frac{1}{2^{2n}} \cdot \frac{2.4.6\ldots 2n}{1.3.5\ldots(2n-1)} \cdot \frac{2n.(2n-1)\ldots(n+2)}{1.2\ldots n-1}$$

$$A_1 = \frac{1}{2^{2n}} \cdot \frac{2.4.6\ldots 2n}{1.3.5\ldots(2n-1)} \cdot \frac{2n.(2n-1)\ldots(n+3)}{1.2\ldots(n-2)}\ \&c.$$

substituting these values for A_0 A_1 &c. in the other terms, and making the proper reductions, we get

$$B_n = \frac{1}{n+1} + \frac{3n}{(n+1).(n+2)}.h + \frac{5n.(n-1)}{(n+1).(n+2).(n+3)}.h^2\ \&c.$$

and taking the coefficient of h^m we have

$$H_n = \frac{(2m+1).n.(n-1)...(n-m+1)}{(n+1).(n+2)...(n+m+1)};$$

and therefore

$$H_0 = 0,\ H_1 = 0 \ldots\ldots H_{m-1} = 0,\ H_m = \frac{(2m+1).m.(m-1)...1}{(m+1).(m+2)...(2m+1)};$$

$$H_{m+1} = H_m \cdot \frac{(m+1)^2}{1.(2m+2)},\ H_{m+2} = H_m \cdot \frac{\{(m+1).(m+2)\}^2}{1.2.(2m+2)(2m+3)}\ \&c.$$

The required formula therefore is

$$\frac{1.2.3...m}{(m+1).(m+2)...2m}\left\{\frac{1.3...(2m-1)}{2.4...2m} \cdot \frac{2^{2m}(\alpha\beta)^m}{(\alpha+\beta)^{2m-1}}\right.$$

$$+ \frac{1.3...(2m+1)}{2.4...(2m+2)} \cdot \frac{(m+1)^2}{(2m+3)} \cdot \frac{2^{2m+2}(\alpha\beta)^{m+1}}{(\alpha+\beta)^{2m+1}}$$

$$\left. + \frac{1.3.5...(2m+3)}{2.4.6...(2m+4)} \cdot \frac{\{(m+1).(m+2)\}^2}{1.2.(2m+2).(2m+3)} \cdot \frac{2^{2m+4}(\alpha\beta)^{m+2}}{(\alpha+\beta)^{2m+3}} + \&c. \right\}.$$

(2) *Method of representing Discontinuous Functions of any number of Breaks.*

21. These particular instances have been put under forms best adapted for many of their applications. We shall now proceed to more general principles, for the representation of discontinuous functions.

To find a formula which shall represent

$$\phi_1(\alpha, \beta, \gamma, \ldots),\ \phi_2(\alpha, \beta, \gamma, \ldots),\ \phi_3(\alpha, \beta, \gamma, \ldots)\ \&c.$$

according as α, β, γ &c. are respectively least.

Denote by ϕ_1, ϕ_2, ϕ_3, &c. the preceding functions, and let the coefficient of $\frac{1}{x}$ in h.l. $\frac{(x+\alpha)(x+\beta)(x+\gamma)\cdots}{x}$ be found, and be represented by S, then shall

$$\phi_1 \frac{dS}{d\alpha} + \phi_2 . \frac{dS}{d\beta} + \phi_3 . \frac{dS}{d\gamma} + \&c.$$

be the required formula.

For when α is the least of the quantities α, β, γ &c. then S simply represents α, and therefore

$$\frac{dS}{d\alpha} = 1, \quad \frac{dS}{d\beta} = 0, \quad \frac{dS}{d\gamma} = 0, \ \&c.$$

and the above formula reduces itself to ϕ_1.

Similarly, when β is least, it becomes $= \phi_2$; and so on.

22. To find a formula, representing a discontinuous function ϕ, which assumes the successive values ϕ_1, ϕ_2, ϕ_3, &c. according as a variable quantity z, commencing with a value $= \alpha$, flows through the successively increasing magnitudes β, γ, δ, &c.

Denote by $S(\alpha, z)$ the coefficient of $\frac{1}{x}$ in h.l. $\frac{(x+\alpha).(x+z)}{x}$,

.....$S(\beta, z)$..... h.l. $\frac{x+\beta).(x+z)}{x}$

&c. &c.

and suppose $\phi = f_1 \frac{dS(\alpha, z)}{d\alpha} + f_2 . \frac{dS(\beta, z)}{d\beta} + f_3 \frac{dS(\gamma, z)}{d\gamma} + \&c.$

Now when $\left.\begin{matrix} z > \alpha \\ < \beta \end{matrix}\right\}$ then $\frac{dS(\alpha, z)}{d\alpha} = 1 \quad \frac{dS(\beta, z)}{d\beta} = 0 \quad \frac{dS(\gamma, z)}{d\gamma} = 0, \ \&c.$

and, by hypothesis, ϕ is then $= \phi_1$.

Again when $x > \beta$ / $< \gamma$ then $\frac{dS(a,x)}{da}=1$ $\frac{dS(\beta,x)}{d\beta}=1$ $\frac{dS\gamma,x)}{d\gamma}=0$, &c.

and ϕ is then $=\phi_2$;

and so on; if we substitute these values in the assumed formula for ϕ, we get successively the Equations,

$\phi_1=f_1$	and therefore $f_1=\phi_1$
$\phi_2=f_1+f_2$	$f_2=\phi_2-\phi_1$
$\phi_3=f_1+f_2+f_3$	$f_3=\phi_3-\phi_2$
&c. = &c.	&c. = &c.

and $\therefore$ $\phi=\phi_1 \frac{dS(a,x)}{da}+(\phi_2-\phi_1)\frac{dS(\beta,x)}{d\beta}+(\phi_3-\phi_3)\frac{dS(\gamma,x)}{d\gamma}+$&c.

Cor. If the function is also discontinuous below a, as x passes through the decreasing magnitudes a, b, c, &c. ϕ assuming then the values ϕ_0, ϕ_{-1} ϕ_{-2}, &c. it is easy to see that then

$$\phi=(\phi_1-\phi_0)\frac{dS(a,x)}{da}+(\phi_2-\phi_1)\frac{dS(\beta,x)}{d\beta}+\&c.$$

$$\left\{+(\phi_0-\phi_{-1})\frac{dS(a,x)}{da}+(\phi_{-1}-\phi_{-2})\frac{dS(b,x)}{db}+\&c.\right.$$

(3) *Geometrical Illustrations of the theory of Discontinuity.* (Vide Pl. 24).

23. It only remains to add a few examples, to illustrate the application of the principles of this Section.

(Fig. 1.) Let *ACa*, *BCb*, be the equal sides, produced, of an isosceles right-angled triangle; we may thus find the Equation of the *black* part *ACB*, as distinguished from that of the *dotted* part *aCb*.

Make the middle point 0 of the hypothenuse, (the length of which we may suppose=2,) the origin of co-ordinates; the axis of x being the hypothenuse itself.

The Equation of *ACa*, generally, is.....$y=1+x$

.....of *BCb*.....is.....$y=1-x$,

and therefore the general equation of the system is

$$(y-x-1)\,(y+x-1)=0.$$

in which, any value of x, as *OP*, corresponds to two values of y as *PQ*, *Pq*; the *least* of these two values is that which belongs to the black part *ACB*, at both sides of the origin; the equation *peculiar* to this part is therefore $y=$ the least of $\begin{cases}1-x\\1+x\end{cases}$ substitute in Art. (16). $1-x$ for α, and $1+x$ for β, and we get for the required equation

$$y=\tfrac{1}{2}(1-x^2)+\frac{1\,.\,1}{2\,.\,4}\,.\,(1-x^2)^2+\frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\,.\,(1-x^2)^3+\&c.$$

We may now treat this value of y, by the ordinary methods of analysis, for the attainment of any object which has sole reference to the division *ACB* of the given system.

Thus to find the area of the triangle *ACB*, we must take as usual $\int_x(y)$ from $x=-1$ to $x=+1$.

Now the general term in the value of y is

$$\frac{1\,.\,1\,.\,3\,.\,5.....(2n-3)}{2\,.\,4\,.\,6\,.\,8.....2n}\,.\,(1-x^2)^n$$

the integral of which from $x=-1$ to $x=+1$ is

$$\frac{2}{(2n-1)\,.\,(2n+1)} \text{ or } \frac{1}{2n-1}-\frac{1}{2n+1};$$

put 1, 2, 3 &c. successively for n in this expression, and we shall generate the series for $f_n(y)$, namely

$$\left.\begin{aligned} &\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\&c.\\ &-\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\&c.\end{aligned}\right\}=1,$$

a result which may easily be verified, by geometrical considerations.

24. We shall terminate this Section with an example, in which the function is, *infinitely*, discontinuous.

Fig. (II). An infinite series of equal and similar arcs of any kind, are ranged at equal distances, across the horizontal right line Ax, the successive arcs being included between pairs of equidistant ordinates; to find the equation of the system.

(Note, the lower branch of Fig. II, only differs from the upper in supposing the extreme points (a_0, b_1) (a_1, b_2) &c. to coincide).

Let $y=f(x)$ be the equation of the curve of which $a_0c_0b_0$ is an arc, x being the abscissa for the curve when continuous, and α the abscissa for the discontinuous curve; let the distance $c_0c_1=1$.

When α is between $-\frac{1}{2}$ and $\frac{1}{2}$; $y=f(\alpha)$ is the equation to $a_0c_0b_0$,

.....$\frac{1}{2}$ and $\frac{3}{2}$; $y=f(\alpha-1)$.....$a_1c_1b_1$,

.....$\frac{3}{2}$ and $\frac{5}{2}$; $y=f(\alpha-2)$.....$a_2c_2b_2$.

Similarly,

when α is between $-\frac{1}{2}$ and $-\frac{3}{2}$; $y=f(\alpha+1)$.....$a_{-1}c_{-1}b_{-1}$

&c. &c. &c.

Hence, if L be the least of the quantities a, $a+1$, $a+2$ &c. $a-1$, $a-2$ &c.

then $y=f(L)$ is the required equation.

Now L is evidently = the least root of the equation $\sin \pi (x-a)=0$.

Hence as in Art. (18) we have

$$y=\text{coefficient of } \frac{1}{x} \text{ in } -f'(x) \text{ h. l. } \frac{\sin \pi (x-a)}{x} \text{ where } f'(x)=\frac{df(x)}{dx};$$

or if we put

$$\frac{\sin \pi (x-a)}{\pi \cos (a\pi)} - x = \phi (x) \text{ then } y=f(0)-f'(0).\phi (0)+\left\{\frac{f'(0).(\phi 0)^2}{1.2}\right\}' \text{ &c.}$$

* Fourier has expressed discontinuous functions by means of Definite Integrals; and when it is unnecessary that their form should be explicit, his results may be conveniently used.

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* Vid. *Fourier, Theorie de la Chaleur.*

SECTION III.

APPLICATION OF THE PRECEDING PRINCIPLES, TO THE PHÆNOMENA OF DEVELOPED ELECTRICITY. (*Plate* 24).

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25. WHEN a body is electrised, either by communication, by influence, or in any other manner, from known or unknown causes, the external action is subject to observation. The law of the action of an electrical particle, at different distances, is at the same time, known to vary as the inverse square of the distance. To find the nature of the electric distribution necessary for the production of any given phænomenon, is evidently a mathematical problem, the solution of which may easily be subjected to the test of experiment. As it is also peculiarly adapted to illustrate the analytical principles which precede, we have taken it as the subject of this Section.

26. (Fig. 3.) For the external action, it will be generally convenient to substitute the electrical tension at the different points (Q) of an infinitely thin, conducting rod (Bx), communicating with the electrised body APB. This tension is the sum of the quotients when we divide the mass of each electrical particle, by the distance of that particle from Q.

27. Suppose the figure APB spherical, and that the electric tension in the rod Ax passing through the centre, at any point Q in the external part is found to be inversely as the distance

OQ from the centre of the sphere; and the law of electric accumulation at the surface of the sphere is required.

Make AB the axis of x, the extremity A of the diameter being the origin; the element of the surface included between two planes at the indefinitely small distance δx, and perpendicular to AB is then $=2\pi a.\delta x$, a being the radius of the sphere; and the distance of any point in this annulus from Q

$$=\{\alpha^2+\beta^2+2\beta.(a-x)\}^{\frac{1}{2}}$$

putting $OQ=\beta$, and if we represent the electric accumulation by A, which is in this case a function of x, the expression for the tension on any point Q of the rod AB is

$$\int_a \frac{2\pi a A}{\{\alpha^2+\beta^2+2\beta.(a-x)\}^{\frac{1}{2}}} \text{ from } x=0 \text{ to } x=2a,$$

neglecting the indefinitely small action due to the electricity on the infinitely small surface of the rod.

Put $x=2at$ to make the limits 0 and 1; the expression is then transformed to

$$4\pi a^2 . \int_t \frac{A}{\{\alpha^2+\beta^2+2a\beta.(1-2t)\}^{\frac{1}{2}}};$$

and therefore by the proposed condition we have

$$\frac{m}{\beta} = \int_t \frac{A}{\{\alpha^2+\beta^2+2a\beta.(1-2t)\}^{\frac{1}{2}}},$$

$$\text{when } a<\beta, \text{ or } \int_t \frac{A}{\left\{1+\frac{2a}{\beta}.(1-2t)+\frac{a^2}{\beta^2}\right\}^{\frac{1}{2}}},$$

equal a constant quantity m, $\frac{a}{\beta}$ being any proper fraction; from

which we get $m = f_t(A)$ (putting $\frac{\alpha}{\beta} = 0$), as also that for internal points, where $\frac{\beta}{\alpha}$ is a proper fraction

$$\frac{m}{\alpha} = \int_t \frac{A}{\{\alpha^2+\beta^2+2\alpha\beta.(1-2t)\}},$$

from which premised observations, the question takes this form:

To find (A), such a function of t, that we may have

$$\int_t \frac{A}{\{\alpha^2+\beta^2+2\alpha\beta.(1-2t)\}^{\frac{1}{2}}} = \text{the least of } \begin{cases} \frac{m}{\alpha} \\ \frac{m}{\beta} \end{cases}.$$

The left-hand member of this equation

$$= \frac{1}{\alpha+\beta} \cdot f_t A \left\{1 - \frac{\alpha\beta.t}{\left(\frac{\alpha+\beta}{2}\right)^2}\right\}^{-\frac{1}{2}},$$

when expanded becomes

$$f_t A \left\{\frac{1}{\alpha+\beta} + \tfrac{1}{2} \cdot \frac{2^2\alpha\beta}{(\alpha+\beta)^3}.t + \frac{1.3}{2.4} \cdot \frac{2^4.\alpha^2\beta^2}{(\alpha+\beta)^5} + \&c.\right\}$$

The right-hand member expanded by Section II, Art. 17, becomes

$$m\left\{\frac{1}{\alpha+\beta} + \frac{1.1}{2.4} \cdot \frac{2^2\alpha\beta}{(\alpha+\beta)^3} + \frac{1.1.3}{2.4.6} \cdot \frac{2^3\alpha^2\beta^2}{(\alpha+\beta)^5} + \&c.\right\},$$

and comparing the general terms in both series, we have

$$f_t A.t^x . \frac{1.3\ldots(2x-1)}{2.4\ldots 2x} . 2^{2x} = m . \frac{1.1.3\ldots(2x-1)}{2.4.6\ldots(2x+2)} . 2^{2x+1},$$

$$\text{or } f_t A.t^x = \frac{m}{x+1},$$

and therefore by the principles of Section I, $A =$ the constant quantity m.

In fact, when the electric accumulation is constant the external attraction will evidently be $\frac{-m}{\beta^2}$; and the integral with respect to β which represents the tension of the fluid at Q is evidently then $= \frac{m}{\beta}$.

28. The law of accumulation in the case considered above is the simplest possible, but when the sphere is subjected to the electric influence of other bodies, the function which expresses that law may have any form (subject to vanishing at infinite distances), the indication of which form is to be had in the law of the attraction on external points, or its integral, the tension of the electricity in the infinitely thin rod ABx.

To generalize the preceding investigation we shall take the following example, which leads to some curious results.

The tension in the external rod, varying as any inverse power of the distance from the centre of the sphere; to find the law of electric accumulation on the spherical surface. (Fig. 3.)

Adopting the same notation as in the last article, and representing the law of external tension by

$m \cdot \frac{a^n}{\beta^{n+1}} \cdot 4\pi a^2$, we have when $a < \beta$ $m \cdot \frac{a^n}{\beta^n} = \int_t \frac{A}{\left\{1 + \frac{2a}{\beta} \cdot (1-2t) + \frac{a^2}{\beta^2}\right\}^{\frac{1}{2}}}$;

and therefore for internal points where $a > \beta$, we may put $\frac{\beta}{a}$ in this expression for $\frac{a}{\beta}$; and thus we find the law of internal tension to be $m \cdot \frac{\beta^n}{a^{n+1}} \cdot 4\pi a^2$: that is, when the external tension

$\propto \frac{1}{\beta^{n+1}}$ the internal tension $\propto \beta^n$; the proposed question now becomes

To find (A) such a function of t that the value of

$$\int_t \frac{A}{\{\alpha^2+\beta^2+2\alpha\beta.(1-2t)\}^{\frac{1}{2}}}, \text{ may be } m.\frac{\alpha^n}{\beta^{n+1}}, \text{ or } m.\frac{\beta^n}{\alpha^{n+1}},$$

according as α is less or greater than β.

The formula which represents the latter discontinuous function (by Section II, Art. 20.) is a series of which the general term is

$$m.\frac{1.3.5...(2x-1)}{2.4.6...2x}.\frac{x.(x-1).(x-2)...(x-n+1)}{(x+1).(x+2).(x+3)...(x+n+1)}.(2n+1).\frac{2^{2x+1}.(\alpha\beta)^x}{(\alpha+\beta)^{2x+1}},$$

and the general term of the integral when we expand the denominator is as before

$$\int_t A.t^x.\frac{1.3.5...(2x-1)}{2.4.6...(2x)}.\frac{2^{2x}.(\alpha\beta)^x}{(\alpha+\beta)^{2x+1}},$$

and if A receive a value such as to make the general terms of both series identical, the series will also be themselves identical.

Equating we get

$$\int_t A.t^x = m.(2n+1).\frac{x.(x-1).(x-2)...(x-n+1)}{(x+1).(x+2).(x+3)...(x+n+1)},$$

from which we may find A, by the method given in Art. 8, Section V. (Vid. Ex. 4.); its value is

$$A = m(2n+1).(-1)^n.\{1-\frac{n}{1}.\frac{n+1}{1}.t+\frac{n.(n-1)}{1.2}.\frac{(n+1).(n+2)}{1.2}.t^2-\&c.\}$$

29. If we make $n=1$, then the law of internal tension is expressed by $4\pi m\beta$, and therefore the internal attraction is the constant quantity $4\pi m$. Hence if a sphere $AEBF$ (Fig. 4) be electrised by the influence of a very distant body P, the force exercised by P to separate the combined electricity in the interior of the sphere may be represented by the constant $-4\pi m$. The law of accumulation is then $A=3m(2t-1)$ which vanishes when $t=\frac{1}{2}$; there is therefore a *transition* line EF, which is a great circle having its plane perpendicular to the direction of P's action; this line divides the sphere into two parts containing equal quantities of the opposite electricities; the maximum accumulation takes place at A and B the poles of the transition circle.

30. The expression for the accumulation A in the general case of Art. 28, may be put under an elegant aud simple form, which facilitates much the *tracing* of the actual distribution. If we observe that

$$1=\frac{d^n.t^n}{1.2.3...ndt^n} \text{ and } (n+1).t=\frac{d^n.t^{n+1}}{1.2...n.dt^n} \text{ \&c. we get}$$

$$A=\frac{m(2n+1)(-1)^n.d^n}{1.2.3...ndt^n}\{t^n-n.t^{n+1}+\frac{n.(n-1)}{1.2}.t^{n+2}-\text{\&c.}\}$$

$$=\frac{m(2n+1)(-1)^n}{1.2.3...n}.\frac{d^n}{dt^n}(tt')^n \text{ putting } t'=1-t.$$

31. To determine the number of transition lines on the surface of the sphere put

$$\frac{1}{1.2...n}.\frac{d^n}{dt^n}(tt')^n=P_n, \text{ then } A=m(2n+1)(-1)^n.P_n;$$

and therefore the equation for determining the transition lines is $P_n=0$.

Now it is evident that

$$1.2.3...(n-1).P_n=\frac{d^{n-1}}{dt^{n-1}}\{(tt')^{n-1}.(1-2t)\}$$

$$=(1-2t).\frac{d^{n-1}}{dt^{n-1}}(tt')^{n-1}-2.(n-1).\frac{d^{n-2}}{dt^{n-2}}.(tt')^{n-1};$$

$$\therefore\ P_n=(1-2t).P_{n-1}-2(n-1).\int P_{n-1}.$$

Suppose now that for a particular value of n, all the roots of the equation $P_{n-1}=0$ are real (as t_1, t_2, t_3,...t_{n-1}); they lie between the roots of the equation $\int P_{n-1}=0$, and from the nature of the question it is obvious that they all lie between 0 and 1; substitute them in the order of their magnitude for t in the above equation, then since P_{n-1} vanishes, the resulting values of P_n will have alternately opposite signs. Again, when $t=0$, $P_n=1=P_{n-1}$; and therefore P_{n-1}, and consequently its integral remains positive from $t=0$ to $t=t_1$ the least root: the order of the signs of $\int P_{n-1}$ is therefore $+$, $-$, $+$, $-$,..... the last in the series being $+$ or $-$ according as n is even or odd, and the corresponding signs of P_n are $-$, $+$, $-$, $+$ &c. in number $(n-1)$ also. If we put $t=0$, P_n is then $=1$, and therefore its sign is $+$; and if we put $t=1$ then $P_n=(-1)^n$; it follows therefore that the substitution of 0, t_1, t_2,...t_{n-1}, 1, in P_n for t gives n alternations of signs, therefore the roots of the equation $P_n=0$ are all real on the supposition that those of $P_{n-1}=0$ are such.

Now by Art. 29, there is one transition line when $n=1$, hence there are two such lines when $n=2$, 3 when $n=3$ &c. so that generally there are n such lines symmetrically situated with respect to the points A and B, (vid. Fig. 5, where the black lines represent the transition circles, which divide the surface of the

sphere into $n+1$ portions, containing alternately the positive and negative electricities).

32. It is obvious that in the $n+1$ portions into which the spherical surface is thus divided, there will be $n-1$ lines in which the accumulation of the respective electricities is a maximum, beside the points A, B; (represented by the dotted lines in Fig. 5), they possess the remarkable property, that with the transition lines, they divide the surface into belts containing alternately exactly equal quantities of the two electricities. Thus the electricities in the portions E_1AF_1, $E_1F_1b_1a_1$ are equal and of opposite kinds, the electricities in $a_1b_1F_2E_2$, and $a_2b_2F_2E_2$ are similarly equal and opposite and so on.

For the whole quantity of electricity, from the point A where $t=0$, to any value of t as t_1

$$= \int_t 4\pi a^2 . A = 4\pi m a^2 (2n+1)(-1)^n \int_t P_n .$$

Now by the theorem of Leibnitz

$$\frac{d^n(uv)}{dt^n} = u\frac{d^n v}{dt^n} + n\frac{dv}{dt}\cdot\frac{d^{n-1}.v}{dt^{n-1}} + \&c.$$

we get $\int_t P_n = \frac{1}{1.2...n}\ \frac{d^{n-1}(tt')^n}{dt^{n-1}}$

$$= t'^n . t - \frac{(n-1)}{1}\cdot\frac{n}{2}\cdot t'^{n-1}t^2 + \frac{(n-1).(n-2)}{1.2}\cdot\frac{n.(n-1)}{2.3}\cdot t'^{n-2}t^3 \ \&c.$$

the integral being supposed to commence from $t=0$.

$$\text{Similarly } \frac{dP_n}{dt} = \frac{1}{1.2...n}\cdot\frac{d^{n+1}(tt')^n}{dt^{n+1}} = -\frac{n+1}{1}\cdot\frac{n}{1}\cdot t'^{n-1}$$

$$+ \frac{(n+1).n}{1.2}\cdot\frac{n.(n-1)}{1}\cdot t'^{n-2}t - \frac{(n+1).n.(n-1)}{1.2.3}\cdot\frac{n(n-1).(n-2)}{1.2}\cdot t'^{n-3}t^2 \ \&c.$$

and comparing both, we get $\int_t P_n = -\frac{tt'}{n.(n+1)}\cdot\frac{dP_n}{dt}$.

Now for all the maximum values of P_n we have $\frac{dP_n}{dt}=0$, hence $\int P_n$ vanishes from $t=0$ to $t=$ any value which makes P_n a maximum, and also between any two such values, from which it is evident that the positive and negative parts of those integrals are exactly equal.

33. Let us now suppose any law of external tension (vanishing at infinite distances) as $\frac{A}{\beta}+\frac{B.\alpha}{\beta^2}+\frac{C.\alpha^2}{\beta^3}+$&c. and seek the law of electric accumulation, necessary to produce it.

It is evident from what has preceded, that the corresponding law of internal tension will be $\frac{A}{\alpha}+\frac{B.\beta}{\alpha^2}+\frac{C.\beta^2}{\alpha^3}+$&c. Now the laws of accumulation necessary to produce the tensions

$$\left(\frac{A}{\beta}\text{ and }\frac{A}{\alpha}\right),\quad \left(\frac{B\alpha}{\beta^2}\text{ and }\frac{B\beta}{\alpha^2}\right),\quad \left(\frac{C\alpha^2}{\beta^3}\text{ and }\frac{C\beta^2}{C\alpha^3}\right)\text{ &c.}$$

are respectively by Art. 30.

$$\frac{A}{4\pi\alpha^2},\quad \frac{-B}{4\pi\alpha^2}.3\frac{d}{dt}.(tt'),\quad \frac{C}{4\pi\alpha^2}\cdot\frac{5}{1.2}\cdot\frac{d^2}{dt^2}(tt')^2,\quad \frac{-D}{4\pi\alpha^2}\cdot\frac{7}{1.2.3}\cdot\frac{d^3}{dt^3}.(tt')^3\text{ &c.}$$

and if we suppose the electrical systems which correspond to these laws of accumulation to be superposed into one system, and the accumulation at each point estimated by the excess of the positive sum above the sum of the negative electricity at that point, it will be the arrangement which was required to produce the given phænomenon.

The accumulation therefore is

$$\frac{1}{4\pi\alpha^2}\left\{A-3B.\frac{d(tt')^2}{dt}+\frac{5C}{1.2}.\frac{d^2(tt')^2}{dt^2}-\text{&c.}\right\},$$

and therefore at similar points in different spheres it is inversely as the surface of the sphere.

34. Let this law of accumulation be expressed by the form $\frac{\phi}{4\pi a^2}$ while the law of external tension is $\frac{1}{\beta}.f\left(\frac{a}{\beta}\right)$.

$$\text{Now } \phi = A - 3B\,.\,\frac{d(tt')}{dt} + \frac{5.C}{1.2}\,.\,\frac{d^2(tt')^2}{dt^2} - \&c.$$

$$= \text{term independent of } \beta \text{ in } \left\{A + B\,.\,\frac{a}{\beta} + C\,.\,\frac{a^2}{\beta^2} + \&c.\right\}$$

$$\times\left\{1 - 3\,\frac{\beta}{a}\,.\,\frac{d(tt')}{dt} + 5\,\frac{\beta^2}{a^2}\,.\,\frac{d^2(tt')^2}{1.2.dt^2}\ \&c.\right\}$$

$$= \text{coefficient of } \frac{1}{\beta} \text{ in } \frac{1}{\beta}\,.\,f\left(\frac{a}{\beta}\right)$$

$$\times\left\{1 - 3\,.\,\frac{\beta}{a}\,.\,\frac{d(tt')}{dt} + 5\,.\,\frac{\beta^2}{a^2}\,.\,\frac{d^2(tt')^2}{1,2.dt^2}\ \&c.\right\}.$$

And if we form the equation $u = t - \frac{\beta}{a}\,.\,u\,.(1-u)$, we get by the theorem of Lagrange

$$u = t - \frac{\beta}{a}\,.\,(tt') + \frac{\beta^2}{a^2}\,.\,\frac{\frac{d}{dt}(tt')^2}{1.2} - \&c.$$

$$\therefore\ 2\,\frac{d}{d\beta}\,.\,(\beta^{\frac{1}{2}}u) = t\beta^{-\frac{1}{2}} - 3\,\frac{\beta^{\frac{1}{2}}}{a}\,.\,(tt') + 5\,.\,\frac{\beta^{\frac{3}{2}}}{a^2}\,.\,\frac{\frac{d}{dt}(tt')^2}{1,2}\ \&c.$$

$$\text{and } \therefore\ \phi = \text{coefficient of } \frac{1}{\beta} \text{ in } \left\{\frac{1}{\beta}\,f\left(\frac{a}{\beta}\right)\right\}\,.\,\frac{d}{dt}\left\{2\beta^{\frac{1}{2}}\,\frac{d}{d\beta}\,.\,(\beta^{\frac{1}{2}}u)\right\}$$

$$= \text{coefficient of } \frac{1}{\beta} \text{ in } \frac{2}{\beta^{\frac{1}{2}}}\,.\,f\left(\frac{a}{\beta}\right)\,.\,\frac{d}{d\beta}\left(\beta^{\frac{1}{2}}\,.\,\frac{du}{dt}\right).$$

But the least root of the equation

$$u = t - \frac{\beta}{\alpha} . u . (1-u) \text{ is}$$

$$u = \frac{\beta+\alpha}{2\beta} - \frac{1}{2\beta} \{\beta^2+\alpha^2+2\alpha\beta . (1-2t)\}^{\frac{1}{2}},$$

$$\text{and therefore } \frac{du}{dt} = \frac{\alpha}{\{\beta^2+\alpha^2+2\alpha\beta . (1-2t)\}^{\frac{1}{2}}},$$

$$\text{and } \frac{d}{d\beta}\left(\beta^{\frac{1}{2}} \frac{du}{dt}\right) = \frac{\alpha}{2\beta^{\frac{1}{2}}} \cdot \frac{\alpha^2-\beta^2}{\{\beta^2+\alpha^2+2\alpha\beta . (1-2t)\}^{\frac{3}{2}}}.$$

$$\text{Hence } \phi = \text{coefficient of } \frac{1}{\beta} \text{ in } \frac{\alpha}{\beta} \cdot f\left(\frac{\alpha}{\beta}\right) \cdot \frac{\alpha^2-\beta^2}{(\beta^2+\alpha^2+2\alpha\beta . (1-2t)^{\frac{3}{2}}}.$$

If therefore we multiply the external tension by

$$\frac{\alpha^2-\beta^2}{\{\beta^2+\alpha^2+2\alpha\beta . (1-2t)\}^{\frac{3}{2}}}$$

and divide by $4\pi\alpha$ the coefficient of $\frac{1}{\beta}$ in the product, the result expresses the law of accumulation.

Or if we multiply the internal tension by the same factor and divide by $\frac{4\pi\alpha^2}{\beta}$ the coefficient of $\frac{1}{\alpha}$ in the product, the result will likewise give the law of accumulation.

This may be expressed in a definite integral by the theorem (*Camb. Trans.* Vol. III, p. 438.)

35. When the electro-motive causes are unknown, then the application of the first of these rules to the observed law of external attraction, or tension (which is the integral of the attraction) will determine the law of distribution. But when the

above causes are known, if f represent the electro-motive force at a distance β from the centre, then $-\int_\beta f$ must be the internal tension produced by the action of the sphere, and the application of the second rule will then give the law of accumulation. We must however always put the internal tension under the form $\frac{1}{a} f\left(\frac{\beta}{a}\right)$.

Thus in Fig. 4. if the point P separates by influence, the combined electricities of the sphere, and Q be a point within at a distance β from the centre, then if we put $OP = ha$ the force on $Q = \frac{c}{(\beta - ha)^2}$ to destroy which the force exerted by the electricity on the surface must $= \frac{-c}{(\beta - ha)^2}$, and therefore in this case

$$\frac{1}{a} f\left(\frac{\beta}{a}\right) = \frac{c}{\beta - ha} = \frac{1}{a} \cdot \frac{c}{\frac{\beta}{a} - h} = \frac{-c}{a}\left\{\frac{1}{h} + \frac{1}{h^2} \cdot \frac{\beta}{a} + \frac{1}{h^3} \cdot \frac{\beta^2}{a^2} \text{ &c.}\right\},$$

and therefore the law of accumulation is

$$\frac{-c}{4\pi a^2}\left\{\frac{1}{h} - \frac{3}{h^2} \cdot \frac{d(tt')}{dt} + \frac{5}{1.2.h^3} \cdot \frac{d^2.(tt')}{dt^2} - \text{&c.}\right\}$$

$$= \frac{ch^{\frac{1}{2}}}{2\pi a^2} \cdot \frac{d}{dh}\left\{\frac{1}{h^{\frac{1}{2}}} - \frac{1}{h^{\frac{3}{2}}} \cdot \frac{d}{dt}(tt') + \frac{1}{1.2.h^{\frac{5}{2}}} \cdot \frac{d^2}{dt^2}(tt')^2 \text{ &c.}\right\}$$

$$= \frac{ch^{\frac{1}{2}}}{2\pi a^2} \cdot \frac{d}{dh}\left(\frac{1}{h^{\frac{1}{2}}} \cdot \frac{du}{dt}\right), \text{ when } u = t - \frac{1}{h} . u . (1-u)$$

$$= \frac{ch^{\frac{1}{2}}}{2\pi a^2} \cdot \frac{d}{dh}\left\{\frac{1}{h^{\frac{1}{2}}} \cdot \frac{h}{\{1 + 2h.(1-2t) + h^2\}^{\frac{1}{2}}}\right\}$$

$$= \frac{c}{4\pi a^2} \cdot \frac{1 - h^2}{\{1 + 2h.(1-2t) + h^2\}^{\frac{1}{2}}} \cdot$$

Using the notation of the last Article, it is evident that the whole quantity of electricity in the sphere, i. e. the difference between the quantities of positive and negative fluids, generally, is equal to $\int_t \phi$ = coefficient of

$$\frac{1}{\beta} \text{ in } \frac{\alpha}{\beta} f\left(\frac{\alpha}{\beta}\right) \cdot \int_t \frac{\alpha^2-\beta^2}{\{\beta^2+\alpha^2+2\alpha\beta.(1-2t)\}^{\frac{1}{2}}}, \text{ from } t=0 \text{ to } t=1.$$

$$\text{Now this integral} = \frac{\alpha^2-\beta^2}{2\alpha\beta} \cdot \left\{\frac{1}{\alpha-\beta} - \frac{1}{\beta+\alpha}\right\} = \frac{1}{\alpha},$$

and therefore the quantity of electricity = coefficient of $\frac{1}{\beta}$ in the external tension, or of $\frac{1}{\alpha}$ in the internal; thus in the example before us this quantity $= -\frac{c}{h}$, but if it were necessary that there should be a given excess E of positive above negative electricity on the surface, we have only to superpose the uniform accumulation $\frac{E+\frac{c}{h}}{4\pi\alpha^2}$ to the variable accumulation expressed by

$$\frac{c}{4\pi\alpha^2} \cdot \frac{(1-h^2)}{\{1+2h(1-2t)+h^2\}^{\frac{1}{2}}}.$$

To find the position of the transition line we must find t in the equation

$$\frac{E}{c} + \frac{1}{h} = \frac{h^2-1}{\{1+2h.(1-2t)+h^2\}^{\frac{1}{2}}}.$$

When the electricity is excited *merely* by influence, we must put $E=0$, the law of accumulation is then as

$$\frac{1}{h} - \frac{h^2-1}{\{1+2h.(1-2t)+h^2\}^{\frac{1}{2}}},$$

that is, the accumulation in any annulus $E'F'$ is proportional of the constant $\frac{1}{PB.PO.PA}$, above the variable $\frac{1}{(PE')^3}$.

36. In general whatever be the surface of revolution, if we put the length of the axis $= 2a$, and $x = 2at$, y will be a given function of t, as well as the annular element of the surface $2\pi y\sqrt{1+\frac{dy^2}{dx^2}}.\delta t$, or $S.\delta t$; the tension in the axis $= \int_t \frac{A.T}{\sqrt{y^2+(\beta-at)^2}}$, and expanding both sides in similar forms and equating we shall have $\int_t AT.P_s.t^s = P_s'$, $P_s t^s$ and P_s' being the general term of both expansions;

$$\therefore \text{ by Section I, } A = \frac{1}{T}.\text{ coefficient of } \frac{1}{x} \text{ in } \frac{P_s'}{P_s}.t^{-1}.$$

The known properties of Laplace's coefficients might have been employed in deducing some of the results given in this Section; but the principal object was to illustrate, in an interesting subject, the application of the *Inverse Calculus of Definite Integrals*.

NOTES.

Note A.—(*Vid. Art.* 9.)

(1.) *Application of the Inverse Method to Logarithmic Functions in general.*

To complete what was observed in Art. 9, let P and Q denote the same quantities as in that Article, and let R be a function of x which remains finite when x is infinite, i. e. of the form

$$A + \frac{B}{x} + \frac{C}{x^2} + \&c.;$$

we are now to find $f(t)$ when $\phi(x) = R \cdot \text{h. l.} \frac{P}{Q}$.

Now by the general theorem (Art. 5.) we have the Equation

$$t f(t) = \text{coefficient of } \frac{1}{x} \text{ in } R \cdot t^{-x} \cdot \text{h. l.} \frac{P}{Q} \ldots\ldots\ldots\ldots (1.)$$

Suppose $R \cdot t^{-x}$ to be actually arranged according to the powers of x, it will consist of two parts, one containing the positive powers of x and the absolute term, all which we shall represent by S; the other, containing the negative powers of x, suppose S'; hence Equation (1) becomes

$$\ldots\ldots t f(t) = \text{coefficient of } \frac{1}{x} \text{ in } (S + S') \text{ h. l.} \frac{P}{Q} \ldots\ldots\ldots\ldots$$

but since h. l. $\frac{P}{Q}$ when expanded contains only negative powers of x like S', we may reject the term S' h. l. $\frac{P}{Q}$ altogether, therefore we need estimate the value of S only, and we shall thence determine

$f(t)$ by the Equation $t f(t) = \text{coefficient of } \frac{1}{x} \text{ in } S \text{ h. l.} \frac{P}{Q} \ldots\ldots\ldots\ldots (2.)$

Now the function $\frac{R}{x}$ is itself of a form proper for $\phi(x)$, let the corresponding value of $f(t)$ be put $= \frac{T}{t}$ when determined by the methods given in Section (1).

$$\text{Hence } T = \text{coefficient of } \frac{1}{x} \text{ in } \frac{R}{x} . t^{-s}$$

$$= \text{term independent of } x \text{ in } Rt^{-s};$$

$$\therefore -\int_t \frac{T}{t} = \text{term independent of } x \text{ in } \frac{R}{x} . t^{-s}$$

$$= \text{coefficient of } x \text{ in } Rt^{-s}.$$

$$\text{Similarly } \int_t \frac{1}{t} \int_t \frac{T}{t} = \text{coefficient of } x^2 \text{ in } Rt^{-s}$$

$$-\int_t \frac{1}{t} \cdot \int_t \frac{1}{t} \cdot \int_t \frac{T}{t} = \text{coefficient of } x^3 \text{ in } Rt^{-s}.$$

$$\&c. = \&c.$$

Having thus obtained the absolute term and the coefficients of x, x^2, x^3, &c. in Rt^{-s}, we have therefore,

$$S = T - x\int_t \frac{T}{t} + x^2 \int_t \frac{1}{t} \int_t \frac{T}{t} - x^3 \int_t \frac{1}{t} \int_t \frac{1}{t} \int_t \frac{T}{t} + \&c. \ldots . (3)$$

all the integrals being made to vanish when $t = 1$, for then $Rt^{-s} = R$, which contains no coefficient of x, x^2, x^3, &c.; the value of T is then = term independent of x in R; that is, A.

By differentiating the Equation (3) and substituting S for its value we get

$$\frac{dS}{dt} = \frac{dT}{dt} - \frac{xS}{t}, \quad \therefore t^s \frac{dS}{dt} + xt^{s-1} . S = t^s \frac{dT}{dt},$$

and integrating we have $St^s = \int t^s \frac{dT}{dt}$, the integral being corrected by the condition $S = A$ when $t = 1$; the value of S is therefore fully discovered.

Recurring now to Equation (2), let $\int_x S$ be represented by $F(-x)$ so that F' denoting the derived function of F, we have $S=-F'(-x)$;

$$\therefore\ tf(t) = -\text{coefficient of } \frac{1}{x} \text{ in } F'(-x).\,\text{h.l.}\,\frac{P}{Q}$$

$$= -\text{coefficient of } \frac{1}{x} \text{ in } F'(-x).\left\{\text{h.l.}\,\frac{P}{x^n} - \text{h.l.}\,\frac{Q}{x^n}\right\},$$

and, by a property of equations demonstrated in my former paper in this Volume (p. 138.), this gives us

$$f(t) = \frac{1}{t}\{F(a_1) + F(a_2) + \ldots + F(a_n) - F(b_1) - F(b_2) \ldots\ldots - F(b_n)\}.$$

If we substitute differential coefficients for integrals, a process similar to that by which S was found, will determine S'; the value is given by the equation $S't^x = -\int t^x \frac{dT}{dt}$, the integral being corrected by the condition, that $S' = R - A$ when $t=1$, and we may verify the results by adding the values of S and S' which will give $S + S' = R.t^{-x}$.

(2.) *On the separation of the positive from the negative powers, of the variable; in functions generally.*

In the preceding example, the form of the function Rt^{-x} contributed materially to facilitate the calculation of the parts which involved the positive and negative powers of x respectively. As on many occasions it would be useful to effect this object, we may here insert a general method founded on very simple principles.

First, let us seek the term independent of h, in a function of h containing both positive and negative powers of h, but composed of a finite number of terms.

Represent by $F(h)$ the given function, and let n be a number greater than any exponent of h, positive or negative which it contains.

Let the nth roots of unity be

$$1,\ a_1,\ a_2, \ldots\ldots a_{n-1} \text{ or } 1,\ e^{\frac{2\pi}{n}\sqrt{-1}},\ e^{\frac{4\pi}{n}\sqrt{-1}} \ldots\ldots e^{\frac{2.(n-1).\pi}{n}\sqrt{-1}}.$$

and suppose $F(h)$ when arranged according to the powers of h to be

$$= A_0 + A_1 h + A_2 h^2 + \ldots + A_{-1} h^{-1} + A_{-2} h^{-2} + \&c.$$

it remains to determine A_0. Putting for h the above roots successively we get

$$F(1) = A_0 + A_1 + A_2 + \ldots + A_{-1} + A_{-2} + \&c.$$

$$F(a_1) = A_0 + A_1 a_1 + A_2 a_1^2 + \ldots + \frac{A_{-1}}{a_1} + \frac{A_{-2}}{a_1^2} + \&c.$$

$$F(a_2) = A_0 + A_1 a_2 + A_2 a_2^2 + \ldots \frac{A_{-1}}{a_2} + \frac{A_{-2}}{a_2^2} + \&c.$$

$$\&c. = \&c.$$

adding all these equations and observing that

$$1 + a_1 + a_2 + \&c. = 0$$

$$1 + a_1^2 + a_2^2 + \&c. = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$1 + \frac{1}{a_1} + \frac{1}{a_2} + \&c. = 0,$$

$$\&c. \qquad \&c.$$

we get $A_0 = \dfrac{F(1) + F(a_1) + F(a_2) + \ldots F(a^{n-1})}{n}$.

Secondly, when the number of terms in $F(h)$ is infinite, we must put $n = \infty$, and observing that generally

$$A_0 = \frac{1}{2\pi\sqrt{-1}} \left\{ F(e^0) . \frac{2\pi\sqrt{-1}}{n} + F\left(e^{\frac{2\pi\sqrt{-1}}{n}}\right) . \frac{2\pi\sqrt{-1}}{n} \right.$$

$$\left. + F\left(e^{\frac{4\pi\sqrt{-1}}{n}}\right) . \frac{2\pi\sqrt{-1}}{n} + \&c. \right\},$$

it follows that when n is infinite

$$A_0 = \frac{1}{2\pi\sqrt{-1}} \int_\theta F(e^\theta) \text{ from } \theta = 0 \text{ to } \theta = 2\pi\sqrt{-1},$$

which accords with the result given in the memoir on the resolution of equations, Section V. in this Volume.

Lastly, we can now determine in any function of x as $f(x)$, that part which contains only positive powers of x, and the absolute term; thus let

$$f(x) = B_0 + B_1 . x + B_2 x^2 + \ldots + B_{-1} . x^{-1} + B_{-2} . x^{-2} + \&c.$$

$$\therefore \frac{f\left(\frac{x}{h}\right)}{1-h} = \{B_0 + B_1 . \frac{x}{h} + B_2 . \frac{x^2}{h^2} + \ldots + B_{-1} . \frac{h}{x} + B_{-2} . \frac{h^2}{x^2} \ldots\} \times \{1 + h + h^2 + \&c.\},$$

and selecting the parts of this product which are independent of h, we get

$$B_0 + B_1 x + B_2 x^2 + \&c. = \text{term independent of } h \text{ in } \frac{f\left(\frac{x}{h}\right)}{1-h}$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_\theta \frac{f(xe^{-\theta})}{1-e^\theta} \text{ from } \theta = 0 \text{ to } \theta = 2\pi\sqrt{-1}.$$

Note B.

(1.) *On the apparently improper Forms of* $\phi(x)$.

IT has been shewn in the first Section, that when $f(t)$ is any of the functions usually received in analysis, $\phi(x)$ must converge to 0 as x approaches ∞, and therefore consists of essentially negative powers of x; it often happens however in the application, that $\phi(x)$ presents itself under apparently a different form. Let us examine the difficulty thus offered; and the simplest mode of doing this, is to take a specific example.

Let $\phi(x) = \frac{1}{x} + \frac{a}{x-1} + \frac{a^2}{x-2} + \ldots + a^{x-1} + a^x \text{h.l.} \frac{a-1}{a}$ to find $f(t)$.

The last terms in $\phi(x)$ seem to be of an improper form as involving positive powers of x, the first terms are also such that $\phi(x)$ apparently becomes infinite for values of x between 0 and ∞. All this however is merely in appearance, for if we put for h.l. $\frac{a-1}{a}$ its value

$$-\frac{1}{a}-\frac{1}{2a^2}-\frac{1}{3a^3}-\ldots-\frac{1}{xa^x}-\frac{1}{(x+1).a^{x+1}}-\frac{1}{(x+2).a^{x+2}}-\&c.$$

we see that the real value of $\phi(x)$ is

$$-\frac{1}{a.(x+1)}-\frac{1}{a^2.(x+2)}-\frac{1}{a^3.(x+3)}-\&c.$$

which is evidently of the proper form, and gives

$$f(t)=-\frac{1}{a}-\frac{t}{a^2}-\frac{t^2}{a^3}-\&c.=\frac{1}{t-a}.$$

A still more striking instance of an apparently improper form of $\phi(x)$ is the following:

Given $\phi(x)=1-x+x.(x-1)-x.(x-1).(x-2)$

$$+\ldots\pm x.(x-1).(x-2).3.2\mp x.(x-1).(x-2)\ldots2.1.\left(1-\frac{1}{\epsilon}\right).$$

This function appears, at first sight, to consist of *only* positive powers of x; while in reality the function consists strictly of negative powers and converges to 0 as x approaches ∞. To see this, put for $\frac{1}{\epsilon}$ or ϵ^{-1}, its value,

$$1-1+\frac{1}{1.2}-\frac{1}{1.2.3}\ldots\pm\frac{1}{1.2\ldots x}\mp\frac{1}{1.2\ldots(x+1)}\ \&c.$$

and we get $\phi(x)$ in the proper form; viz.

$$\phi(x)=\frac{1}{x+1}-\frac{1}{(x+1).(x+2)}+\frac{1}{(x+1).(x+2).(x+3)}-\&c.$$

and therefore by Art. 8. Ex. 5.

$$f(t)=1-(1-t)+\frac{(1-t)^2}{1.2}-\frac{(1-t)^3}{1.2.3}+\&c.=t^{t-1}.$$

Consider lastly the case of $\phi(x)=(a+x)^{\frac{1}{2}}-(x-a)^{\frac{1}{2}}$, to find $f(t)$.

Observing that $(x-a)_{\frac{1}{2}}=(a+x)^{\frac{1}{2}}.\left(1-\frac{2a}{a+x}\right)^{\frac{1}{2}}$ and expanding, we have

$$\phi(x)=\frac{a}{(a+x)^{\frac{1}{2}}}+\frac{1.1}{1.2}\cdot\frac{a^2}{(a+x)^{\frac{3}{2}}}+\frac{1.1.3}{1.2.3}\cdot\frac{a^3}{(a+x)^{\frac{5}{2}}}+\&c.$$

and therefore by Art. 12. Section I.

$$f(t)=t^{a-1}\left\{\frac{a\,(\text{h. l. }t)^{-\frac{1}{2}}}{\sqrt{\pi}\,(\text{h. l. }t)^{-\frac{1}{2}}}+\frac{1.1}{1.2}\cdot\frac{a^2.(\text{h. l. }t)^{\frac{1}{2}}}{\sqrt{\pi}\,(\text{h. l. }t)^{\frac{1}{2}}}+\frac{1.1.3}{1.2.3}\cdot\frac{a^3.(\text{h. l. }t)^{\frac{3}{2}}}{\sqrt{\pi}\,(\text{h. l. }t)^{\frac{3}{2}}}\ \&c.\right\}$$

$$=\frac{t^{a-1}}{2\sqrt{\pi}}\left\{\frac{2a\,(\text{h. l. }t)^{-\frac{1}{2}}}{1}-\frac{(2a)^2.(\text{h. l. }t)^{\frac{1}{2}}}{1.2}+\frac{(2a)^3.(\text{h. l. }t)^{\frac{3}{2}}}{1.2.3}-\&c.\right\}$$

$$=\frac{t^{a-1}}{2\pi^{\frac{1}{2}}.(\text{h. l. }t)^{\frac{1}{2}}}\cdot\left\{1-t^{-2a}\right\}$$

$$=\frac{1}{2\pi^{\frac{1}{2}}}\cdot\frac{t^a-t^{-a}}{t\,(\text{h. l. }t)^{\frac{1}{2}}}.$$

(2.) *Method of estimating the Values of Operative Functions.*

If any function of x be reduced to the form of a sum, composed of terms which are other functions of x; the identity thus formed will continue to subsist, when x is changed into $x+h$ in both its members. Let the operation which indicates that x is to be changed to $x+h$ be denoted by ψ, let ψ^n denote that x must be changed to $x+nh$ and $\psi^{\frac{1}{n}}$, into $x+\frac{h}{n}$, then ψ^n is evidently equivalent to the repetition n terms of the operation ψ, while the latter is equivalent to the repetition n terms of $\psi^{\frac{1}{n}}$; if beside the operation ψ^n the function of x is also to be

multiplied by a constant quantity A_n, we may represent the combined operation by $A_n\psi^n$, and the sum of any number of such operations as

$$A_m\psi^m + A_n\psi^n + A_r\psi^r + \&c.$$

may be represented by the operative function $F(\psi)$.

Suppose also that $\phi(x)$ the given function of x which is subjected to the above operations, is by the inverse calculus reduced to the form $\int f(t).t^x$; the result of the operation $F(\psi)$ will be the sum of the partial results of the same operation on each element $f(t).t^x.\delta(t)$;

$$\text{but } \psi^n(t^x) = t^{x+nh} = (t^h)^n.(t^x);$$

$$\therefore F(\psi).t^x = F(t^h).t^x;$$

$$\therefore F(\psi).\phi(x) = \int f(t).F(t^h).t^x \ldots\ldots\ldots\ldots (1.)$$

Let the operation which is the inverse of $F(\psi)$ be represented by $F^{-1}(\psi)$ so that the latter operation performed with the former neutralizes it, then

$$F^{-1}(\psi).\{Ce^{mx}\} = C.F^{-1}(e^{mh}).e^{mx};$$

let the roots of the

$$F^{-1}.(e^{mh}) = 0 \text{ be } m_1,\ m_2,\ m_3 \ldots\ldots\ldots \&c.;$$

$$\therefore F^{-1}(\psi)\{A_1e^{m_1x} + A_2e^{m_2x} + A_3e^{m_3x} \&c.\} = 0, \text{ where } A_1,\ A_2,\ A_3 \&c.$$

are arbitrary constants;

$$\therefore F(\psi)(0) = A_1e^{m_1x} + A_2e^{m_2x} + \&c.;$$

which appendage must be added to the operation $F(\psi)\phi(x)$, since $\phi(x)$ is analytically to be treated as $\phi(x) + 0$.

It is evident that the operation ψ^0 is equivalent to multiplying the given function by unity, and therefore $\psi - \psi^0$ is equivalent to the operation Δ of finite differences; consequently ψ is equivalent to $\Delta + 1$; Hence

$$F(\Delta).\phi(x) = \int F(t^h - 1).f(t).t^x \ldots\ldots\ldots\ldots (2);$$

writing $\frac{\Delta}{h}$ for Δ in this equation, and making h indefinitely small, when $\frac{\Delta}{h}$ is usually represented by $\frac{d}{dx}$, and $\frac{t^h-1}{h} = \text{h. l. } (t)$ we get

$$F\left(\frac{d}{dx}\right) . \phi(x) = \int_i f(t) . F(\text{h. l. } t) . t^x \ldots\ldots\ldots\ldots (3),$$

where the appendage is, as before, to be annexed.

To reduce functions containing positive power of x, and zero to the form $\int_i f(t) . t^x$ belongs to the next part; but we may here indicate, that the series given for discontinuous functions, may be used in this case, to reduce the proposed functions to forms to which the principles of this part may be applied.

It was necessary to make the preceding remarks on Operative Functions, to explain more fully the allusions in reference to this subject in the Introduction.

CAIUS COLLEGE,
May 26, 1832.

R. MURPHY.

XIV. *On the Phænomena of Newton's Rings when formed between two transparent Substances of different refractive Powers.*

By G. B. AIRY, M.A. F.R.A.S. F.G.S.

LATE FELLOW OF TRINITY COLLEGE, AND PLUMIAN PROFESSOR OF ASTRONOMY AND EXPERIMENTAL PHILOSOPHY IN THE UNIVERSITY OF CAMBRIDGE.

[Read *March* 19, 1832.]

In a paper communicated to this Society about four months since, I stated my expectation (founded on Fresnel's theory) that if a lens of a low-refracting substance were placed on a plane surface of a high-refracting substance, and if light polarized in the plane perpendicular to the plane of reflection were incident upon it, then so long as the angle of incidence was less than the polarizing angle of the low-refracting substance, or greater than that of the high-refracting substance, Newton's rings would be seen with a black center; but if the angle of incidence was greater than the first of these and less than the second, Newton's rings would be seen with a bright center. I have now to announce the fulfilment of this anticipation.

Before describing the method by which I have succeeded in the examination of these phænomena, I think it right to give

a theoretical calculation of the intensity of light in the rings: as without this the necessity for some of the precautions will not be sufficiently evident.

Conceive two nearly parallel plates of different media to be separated by a plate of air whose thickness is T; and let the vibration in the plane of reflexion, of an incident stream of light within the first medium, be represented by $a \sin \frac{2\pi}{\lambda}(vt-x)$ where x is the equivalent in air to the actual distance of a particle from some fixed point, (the light being supposed polarized in a plane perpendicular to the plane of reflexion). Let ι be the angle of incidence on the last surface of the first medium; ι' the angle of refraction, which is the same as the angle of incidence on the first surface of the second medium; and ι'' the angle of refraction in the second medium. A part of the light will be reflected at the last surface of the first medium: a part will reach the first surface of the second medium, where it will be subdivided; and one portion will be reflected to the surface of the first medium where it will be again divided, and one of its parts will enter in the same direction as that which was reflected at first. In this the phase of the undulation will be *behind* that which was first reflected by the quantity corresponding to the space $2T\cos\iota'$: or if $\frac{2\pi}{\lambda}(vt-x)$ be still taken as the measure of the phase of the ray first reflected, $\frac{2\pi}{\lambda}(vt-x)-\frac{4\pi}{\lambda}T\cos\iota'$ will be that of the ray which has been reflected at the surface of the second medium and then enters the first. The quantity $\frac{4\pi}{\lambda}T\cos\iota'$ we shall for abbreviation call V. Of the light which reaches the surface of the first medium, a part will be partially reflected at the surface

of the second medium, and will partially enter the first medium: its phase will be $\frac{2\pi}{\lambda}(vt-x)-2V$; and so for succeeding reflexions.

Now suppose that at the last surface of the first medium, the coefficient of the incident vibration being 1, that of the reflected vibration is e and that of the refracted f; at the first surface of the second medium, suppose the coefficient of the reflected vibration to be g; and for light incident from air on the surface of the first medium, suppose the coefficients of the reflected and refracted vibrations to be h and k. Then, the coefficient in the incident light being a,

That in the first reflected light is.....ae

that in the refracted light is af

that in the light reflected at the second medium is afg

and that in the light refracted into the first medium is.....$afgk$

that in the light reflected from the first medium is $afgh$

that in the light reflected from the second medium is afg^2h

and that in the light refracted into the first medium is.....afg^2hk

and so on; the coefficients after the first following a geometrical progression whose ratio is gh. Thus it appears that the whole vibration will be

$$a.e.\sin\frac{2\pi}{\lambda}(vt-x)+a.fgk\left\{\sin\left(\frac{2\pi}{\lambda}(vt-x)-V\right)+gk.\sin\left(\frac{2\pi}{\lambda}(vt-x)-2V\right)+\&c.\right\},$$

$$\text{or } a.e.\sin\frac{2\pi}{\lambda}(vt-x)+a.fgk.\frac{\sin\left(\frac{2\pi}{\lambda}(vt-x)-V\right)-gh.\sin\left(\frac{2\pi}{\lambda}(vt-x)\right)}{1-2gh.\cos V+g^2h^2}.$$

Now in Fresnel's expressions,

$$e=\frac{\tan(\iota-\iota')}{\tan(\iota+\iota')},$$

$$f=\frac{\cos\iota}{\cos\iota'}\left(1-\frac{\tan(\iota-\iota')}{\tan(\iota+\iota')}\right),$$

$$g=\frac{\tan(\iota'-\iota'')}{\tan(\iota'+\iota'')},$$

$$h=\frac{\tan(\iota'-\iota)}{\tan(\iota'+\iota)},$$

$$k=\frac{\cos\iota'}{\cos\iota}\left(1-\frac{\tan(\iota'-\iota)}{\tan(\iota'+\iota)}\right).$$

Hence $fk=1-e^2$, and $gh=-ge$; and the expression becomes

$$ae\sin\frac{2\pi}{\lambda}(vt-x)+ag(1-e^2).\frac{\sin\left(\frac{2\pi}{\lambda}(vt-x)-V\right)+ge.\sin\left(\frac{2\pi}{\lambda}(vt-x)\right)}{1+2ge\cos V+g^2e^2}.$$

Resolving this into the form

$$P\sin\frac{2\pi}{\lambda}(vt-x)+Q\cos\frac{2\pi}{\lambda}(vt-x),$$

the intensity or P^2+Q^2 becomes

$$a^2\frac{g^2+e^2+2ge\cos V}{1+2ge\cos V+g^2e^2}.$$

The maxima and minima of this correspond to the maxima and minima, or the contrary, of $\cos V$. When $V=0$, 2π, &c. that is when $T=0$, or $=\frac{\lambda}{2\cos\iota'}$, or $=\frac{2\lambda}{2\cos\iota'}$, &c. the intensity of the reflected light is

$$a^2\left(\frac{g+e}{1+ge}\right)^2,$$

and when $T = \frac{\lambda}{4\cos\iota}$, $\frac{3\lambda}{4\cos\iota}$, &c. the intensity is

$$a^2\left(\frac{g-e}{1-ge}\right)^2,$$

and the excess of the latter above the former is

$$-a^2.\frac{4eg(1-e^2)(1-g^2)}{(1-e^2g^2)^2}.$$

This is the difference of intensity of the brightest and of the darkest parts of the rings: and when it is positive, the center of the rings is dark.

Now $\tan^2(\iota+\iota')$ is always greater than $\tan^2(\iota-\iota')$, and $\tan^2(\iota'+\iota'')$ is always greater than $\tan^2(\iota'-\iota'')$: so that $(1-e^2).(1-g^2)$ is always positive. Consequently the central spot is black when e and g have different signs, and bright when they have the same sign. Or as $\tan(\iota-\iota')$ is always negative and $\tan(\iota'-\iota'')$ always positive, the central spot is black when $\tan(\iota+\iota')$ and $\tan(\iota'+\iota'')$ have the same sign, and bright when they have different signs: that is, it is dark when $\iota+\iota'$ and $\iota'+\iota''$ are both less or both greater than 90°, and bright when $\iota+\iota'$ is less than 90° and $\iota'+\iota''$ greater than 90° (or vice versâ). From this it follows that while the angle of incidence is less than the polarizing angle of the first medium, the central spot is black: at that polarizing angle the rings disappear (as $e=0$): from that angle to the polarizing angle of the second medium the central spot is bright: at the polarizing angle of the second medium the rings disappear (as $g=0$); and beyond that, the central spot is again dark.

Now let us estimate the intensity of the light at the central spot when the first ring is black (the angle of incidence being between the two angles of polarization). If the first ring is black

we have $\frac{g-e}{1-ge}=0$, whence $g=e$: and the intensity in the central spot becomes $a^2.\left(\frac{2e}{1+e^2}\right)^2$. The condition $g=e$ gives

$$\frac{\tan(\iota'-\iota'')}{\tan(\iota'+\iota'')}=\frac{\tan(\iota-\iota')}{\tan(\iota+\iota')},$$

whence $\sin^2 2\iota'=\sin 2\iota.\sin 2\iota''$:

$$\text{or } \cos^2\iota'=\frac{1}{mm'}\cos\iota.\cos\iota''.$$

where m and m' are the refractive indices of the two media. Without attempting to solve this equation generally, suppose $m=1{,}53$ and $m'=2{,}45$ (which correspond nearly to plate glass and diamond). The values of ι' at the polarizing angles are $56°.49'.54''$ and $67°.47'.48''$; and the value of ι' which makes the first ring black is $63°.19'.4''$; the values of ι and ι'' corresponding to this are $35°.45'.57''$ and $21°.23'.21''$: whence $e=g=0{,}083215$; and the intensity of the light at the central spot $=a^2\times 0{,}02732$.

But to obtain a practical idea of the import of this expression we must compare it with the intensity of light in the rings in some other position. Now when the incidence is perpendicular, the expressions above give for the difference of the light in the dark spot and bright rings, $a^2\times 0{,}28159$. Consequently the intensity of light in the rings seen between the two polarizing angles is less than one-tenth of that in the rings seen at a nearly perpendicular incidence. As the latter are by no means vivid, we must expect the former to be faint.

The intensity of the rings which would be produced at the same angle of incidence by light polarized in the plane of reflection, found in the same way, $\left(\text{putting } e'=\frac{\sin(\iota-\iota')}{\sin(\iota+\iota')} \text{ and } g'=\frac{\sin(\iota'-\iota'')}{\sin(\iota'+\iota'')}\right)$

is $a^2 \times 0{,}66487$; and is consequently about twenty-four times greater than that of the rings of which we are treating.

This shews that much care will be necessary to make the rings visible. Suppose for instance that the incident light is polarized by a plate of tourmaline, or (which amounts to the same thing) that the reflected light is examined by a tourmaline, with its axis perpendicular to the plane of reflection. Few tourmalines are so perfect as to transmit no more than one twenty-fourth part of the light polarized perpendicular to their axis. If then the rings are examined with one of these, the rings of which we are in quest (whose center is bright) will be mixed with rings produced by light polarized in the plane of reflexion (whose center is black) of at least equal intensity: and their character will therefore be entirely destroyed. If instead of a tourmaline we use a doubly-refracting prism, with which both sets of rings are exhibited, separated from each other, there will be no fear of confusion of the rings, but a sheet of bright light (from the rays polarized in the plane of reflection) will be spread over the faint rings that we are seeking, and will effectually make them invisible.

The plan which I have successfully adopted is, to combine a tourmaline and doubly-refracting prism. By means of the tourmaline (with axis perpendicular to the plane of reflection) the brightness of the sheet of light, which would otherwise cover the rings that we have to examine, is so far diminished, that it offers no serious obstacle. At the same time the other set of rings is seen, and serves very well as an object of comparison.

To destroy the reflection at the upper surface of the imposed lens is a matter of importance. I have used a plano-convex lens of 5,8 inches focal length with an obtuse-angled prism placed

upon its plane side, the obtuse angle being over the center of the lens. A drop of water was placed between them. Though its refractive index differs sensibly from that of the glass, yet the reflection at the common surface of the prism and lens is almost totally destroyed, for the following reason. The surface of the lens is I suppose very slightly convex, and when the drop of water is interposed, and the air-bubbles are rubbed out, Newton's rings are seen, very large though slightly irregular, with the black spot in the center. The rings in question are seen through this black spot, and consequently are not injured by the effects of reflection. The water seems to have the power of bringing the lens and prism into closer contact than is otherwise attainable:* for I am well convinced that no force that could be applied without injuring them would bring them so near together as to exhibit the central black.

For the denser medium I have used a diamond with a surface of about $\frac{1}{12}$ inch in diameter, mounted in a ring: for the use of which I am indebted to the politeness of William John Broderip, Esq. Vice President of the Geological Society. When the lens and prism were placed on this, a small system of rings was seen perfectly distinct and well formed, the diameter of the fifth ring not exceeding $\frac{1}{2}$ of the diameter of the surface.

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* I may here mention a curious circumstance which occurred to me in the use of this combination. After leaving the prism, with the lens hanging to its lower surface, for one or two days, the water contracted itself to a spot (having partly gone off, I suppose, by evaporation) of about $\frac{1}{4}$ inch in diameter, its outline following most accurately the course of one of the rings (I think the third) even in its deviations from symmetry. In this state I was not able to move the lens upon the prism, though I applied a force parallel to the surface of the prism sufficiently great to shiver large splinters from the lens. On dipping them into water they instantly dropped asunder.

These rings were examined with the combination of tourmaline and doubly refracting prism that I have described. When the angle of incidence was small, the rings formed by light polarized perpendicular to the plane of reflection were seen sufficiently vivid, with black center, accompanied by the other set of rings which were faint. When the angle of incidence reached the polarizing angle of the glass, the first set of rings disappeared. On increasing the angle, the first set of rings was again seen with center white. In the most favourable state, the first set of rings was much more faint than the second, but not so faint that there could be the slightest doubt upon the fact of the existence of the rings and the whiteness of the center, as I saw them repeatedly with every change in the arrangement of the apparatus, and saw a succession of several rings. The white spot appeared larger than the dark spot in the other set of rings, but this I imagine is owing merely to the undefined nature of the spots, and to the circumstance that, in appreciating their comparative extent, the eye always gives credit to the brightness for a greater surface than it can properly claim. In respect of dimensions of corresponding parts, I could see no difference. On increasing the angle of incidence, the first set of rings again disappeared, and reappeared in great brilliancy, the center being now black.

I am willing to think these experiments important, because they bear immediately upon a part of Fresnel's theory which has always appeared to me most liable to objection, namely the formulæ for the extent of vibration in reflected and refracted rays. On the truth of Fresnel's general theory as a mere geometrical representation, namely that light consists of transversal vibrations, and that polarized light is light in which all the

vibrations are perpendicular to the plane of polarization, I shall say nothing, because I do not think it will be doubted by any one who is well acquainted with the experiments and has examined their agreement with calculation. But on the theorems for intensity in reflected rays, &c., involving points of the greatest obscurity, and supported only by very forced suppositions, any one may I think with reason be sceptical. The phænomena described here and those described in a former paper (On a remarkable modification, &c.) depend entirely, in theory, upon the changes of sign of certain quantities which enter into Fresnel's expressions for these intensities. With respect to the absolute measure of the intensities I can say nothing, except that the general appearance of the brightness is sufficiently in accordance with the law. On the whole I think that these experiments give great probability to the truth of the formula considered as a general law: and that they establish with certainty that part of it which implies that, after passing a certain angle, the direction of the vibration in the reflected ray (considered with respect to that in the incident ray) is reversed.

OBSERVATORY,
Feb. 4, 1832.

G. B. AIRY.

POSTSCRIPT.

SINCE the above account was written I have (with a favourable sky) seen the white-centered rings many times, and several times with a doubly-refracting prism only, unassisted by a tourmaline. In examining one part of the phænomena I find that there is a discordance of a most curious kind from what the strict theory had led me to expect.

When the light is incident at the polarizing angle of the glass, the rings, so far as I can see, vanish totally. Though I have looked several times with the most scrutinizing attention, I have not been able to see the least trace. If the angle of incidence is gradually increased till it exceeds the polarizing angle, the black-centered rings disappear gradually without altering their size (a considerable quantity of light being still reflected from the diamond) and white-centered rings of the same size appear in their place, without any intermediate stage except a total absence of rings. From the agreement of this with theory I conclude that the polarization of light at the inner surface of glass is (to the senses) complete. But at the polarizing angle of the diamond the case is perfectly different. On increasing the angle of incidence till it exceeds this angle, the white-centered rings do not disappear, but the first black ring contracts so as to leave no central white, and becomes itself the black center. After this there is no material change: I find however that the black center of the rings produced by light polarized perpendicular to the plane of reflection is always (beyond the polarizing angle of the diamond) sensibly larger than the black center of the rings produced by light polarized in the plane of reflection.

The nature of this transition from rings of one character to rings of the opposite character appears to me to be, theoretically, extremely curious. As the rings do not disappear, it is plain that if light polarized perpendicular to the plane of incidence (or whose vibrations are entirely in that plane) is incident at what is called the maximum polarizing angle of the diamond, a portion of it is still reflected. Still however on increasing the angle of incidence the character of the rings is changed: and

this takes place at an angle where (so far as we are entitled to conclude) there is nothing peculiar in the reflexion from the glass; and we are therefore compelled to admit that, the incident vibration being $a \cdot \sin \frac{2\pi}{\lambda}(vt-x)$, when the angle of incidence is increased so as to exceed that angle the reflected vibration is changed from $+p \cdot \sin \frac{2\pi}{\lambda}(vt-x)$ to $-q \cdot \sin \frac{2\pi}{\lambda}(vt-x)$. A similar change takes place at the polarizing angle of the glass: but there, as we have seen, the transition from $+p$ to $-q$ is effected by passing through 0, or by the entire cessation of reflection at one angle of incidence; which is not the case at the polarizing angle of the diamond. How then is the gradual change from $+p \sin \frac{2\pi}{\lambda}(vt-x)$ to $-q \cdot \sin \frac{2\pi}{\lambda}(vt-x)$ to be explained? I answer that the phœnomena prove that it follows from a *gradual change of phase*, while the coefficient is not much altered. In other words (neglecting the trifling alteration in the coefficient) the quantity $+p \sin \frac{2\pi}{\lambda}(vt-x)$ is changed to $-p \sin \frac{2\pi}{\lambda}(vt-x)$, not by the disappearance of p, but by the expression assuming the form $p \sin \left\{\frac{2\pi}{\lambda}(vt-x)-\theta\right\}$, where θ increases from 0 to π. This may be popularly explained in the following manner. The common Newton's rings, formed between two lenses, are produced by the interference of the light reflected from the lower surface of the upper lens with that reflected from the upper surface of the lower lens. Now if the upper lens be raised a little, or the lower depressed a little, the rings contract. As the only immediate effect of depressing the lower lens is to cause the light reflected from it to describe a longer path, or to have its phases retarded, it appears that a contraction of the rings may

be considered as the effect of a retardation in the phase of the light reflected from the lower surface. The contraction of the rings then in passing the polarizing angle of the diamond requires us to admit that the phase of the reflected light (the incident light being polarized perpendicular to the plane of the reflexion) is, on increasing the angle of incidence by a few degrees, retarded nearly 180°.

The retardation however is not quite 180°. For if it were, the character of the rings would be exactly changed, so that the proportion of the size of the central black spot to that of the first white ring would be the same as that of the central white spot (before the change) to the first black ring. But as the central black spot formed by rays polarized perpendicular to the plane of reflexion is distinctly larger than that formed by rays polarized in the plane of reflexion, it seems that the black ring has not contracted completely, or that the alteration of phase is not quite 180°. This reasoning it must be confessed is not certain, as the same thing would be explained by supposing a small alteration of phase in the light polarized in the plane of reflexion. I may mention here that in the Newton's rings formed between two lenses of the same kind of glass, the central black spot in those formed by light polarized perpendicular to the plane of reflexion is larger than in those formed by the light polarized in the plane of reflexion.

If, while the white-centered rings are under examination, the tourmaline and doubly-refracting prism are turned round, the rings become faint, but do not disappear, and are changed into black-centered rings by the contraction of the rings. This is exactly similar to what takes place when a lens is placed on a metallic surface, and it proves that (as in the former paper),

while the angle of incidence is a few degrees less than the maximum polarizing angle of the diamond, the phase of light polarized perpendicular to the plane of reflexion is more retarded than the phase of light polarized in that plane.

I have not found any variation in these results from changing the position of the plane of reflexion on the diamond surface.

The result of these experiments and reasonings may be thus stated.

1. When the angle of incidence is less than the maximum polarizing angle of the diamond, the nature of its reflexion is similar to that of metallic reflexion: the phase of vibrations in the plane of reflexion being more retarded than that of vibrations perpendicular to the plane of reflexion, but perhaps by a smaller quantity than in reflexion from metals.

2. In the neighbourhood of the polarizing angle, the nature of the reflexion is different from any that has hitherto been described. The vibrations in the plane of reflexion do not vanish, but on increasing the angle of incidence by three or four degrees the phase of vibration is gradually retarded by nearly 180°. In the reflexion of light whose vibrations are perpendicular to the plane of reflexion there is no striking difference between the effects of diamond and those of glass.

3. For angles of incidence greater than the polarizing angle, there is no sensible difference between the effects of diamond and those of glass.

I may remark that the extent of vibration in the plane of reflexion may be represented thus (the formula being purely empirical and given only for illustration). The vibration in the incident light being $a \sin \frac{2\pi}{\lambda}(vt-x)$ that in the reflected light is

$$\frac{\tan(\iota'-\iota'')}{\tan(\iota'+\iota'')}\, a \sin\frac{2\pi}{\lambda}(vt-x) - ba\cos\frac{2\pi}{\lambda}(vt-x)\,,$$

where b is always small but never $=0$, and is perhaps constant.

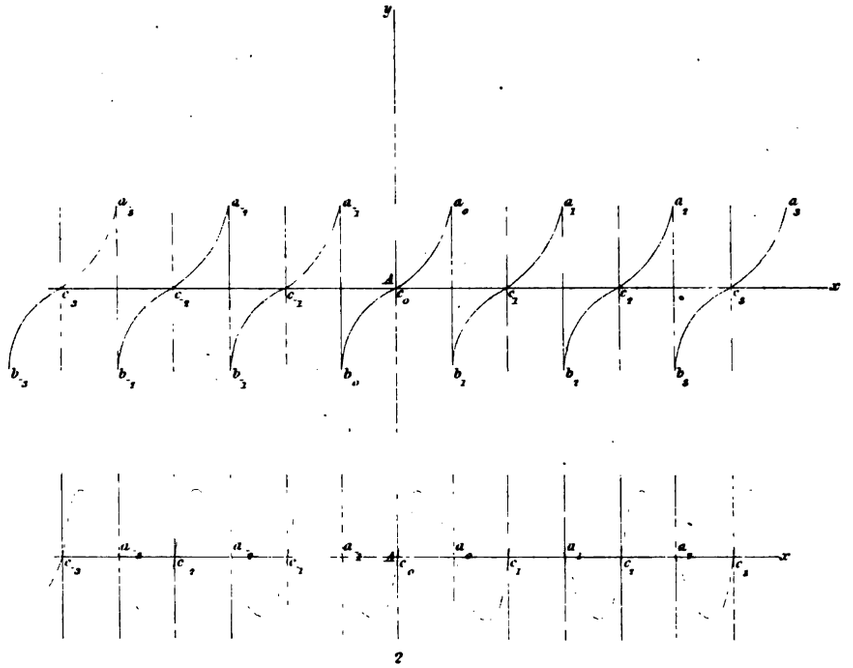
The conclusions at which I have arrived are at variance with one of Sir David Brewster's (Phil. Trans. 1815). Sir David Brewster's character as an experimental philosopher stands deservedly so high, and my estimation of his accuracy, (as observed by myself in the repetition of many of his experiments) is so great, that I think it necessary to point out distinctly the nature of this disagreement.

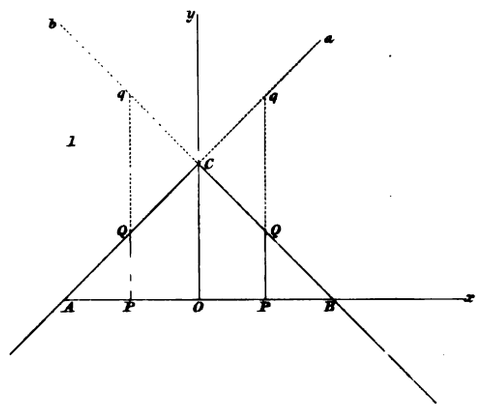
Sir David Brewster states that homogeneous light is completely polarized by the diamond at the proper angle. I have made no experiments here with homogeneous light, and I know that, on account of its extreme faintness however obtained, little confidence can be placed in results which depend only on the evanescence of the reflected light. But the phænomena observed by me are entirely inconsistent with this supposition. If homogeneous light were used, then (on this supposition) the bright-centered rings would disappear and black centered rings would succeed them as at the polarizing angle of the glass. If white light were used, the rings in the neighbourhood of the polarizing angle would be wholly coloured, and on changing the angle the intensity of the different colours in each ring would alter, but there would be nothing like contraction. Thus at a certain angle the brightest part of the red would be at the center of the spot, and its faintest part would be in the first ring; while for the blue the places would be reversed: on increasing the angle the brightest parts of both would be in the first ring. Whereas in my experiments there was no discoverable alteration in the colours of the rings, there never was seen a bright red center

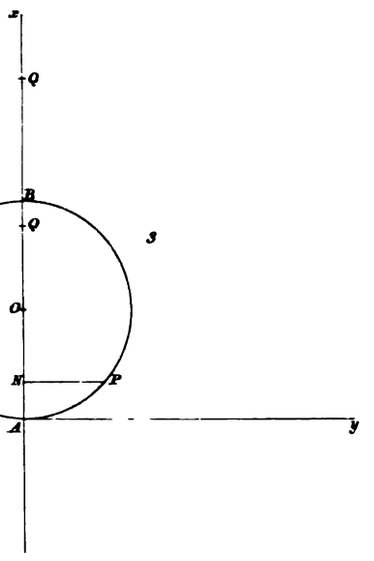
surrounded by a bright blue ring; but the rings, without changing their character as to colour, diminished steadily till the central spot was as it were squeezed out. Whether the only diamond which I have used may possess any peculiarity which distinguishes it from those used by Sir David Brewster I cannot say. Meantime I may observe that the singularity in the reflexion at the surface of the diamond makes it not improbable that there may be some singularity in the refraction also, and renders a more extended inquiry into the laws both of its reflexion and of its refraction highly desirable.

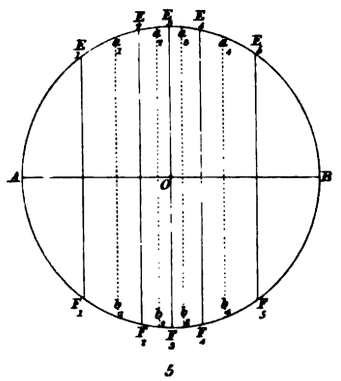
OBSERVATORY,
Feb. 16, 1832.

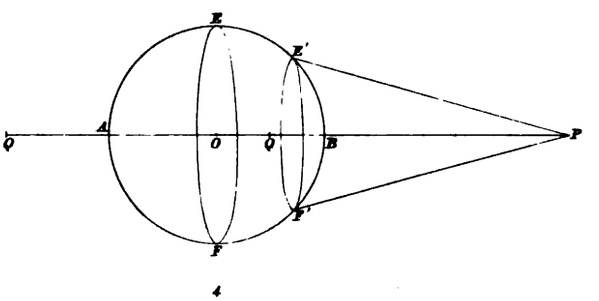
G. B. AIRY.











Josiah Neele Sculp. 352 Strand.

XV. *Description of a Machine for resolving by Inspection certain important Forms of Transcendental Equations.*

By Sir J. F. W. HERSCHEL,

MEMBER OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

[Read *May* 7, 1832.]

(1.) In the course of a conversation with Mr. Babbage on the subject of applying machinery to the performance of numerical computations it occurred to me that, seeing the perfection with which every description of wheelwork and rectilinear or parallel motion can now be executed, almost any combination of circular functions involving as well the arc itself and its multiples and submultiples, as their sines, cosines, chords, &c. *might* be represented by the motion of a point, or by the difference of motions of two points, regulated by mechanism, with almost perfect precision, and that it would therefore need nothing more than to mark the arrival of such a point at some definite line or circle, or to cause the machine in some way or other to come to rest when such difference should attain a given magnitude, or when any other assigned condition should be fulfilled, to obtain a solution of the equation expressive of that condition; which solution should be limited only, in point of exactness, by the precision of the workmanship and the accuracy attainable in hitting the coincidence and reading off

the result. This idea (the principle of which, it will be observed, is entirely independent of all numerical calculations, or of any application of wheelwork to perform such calculations, and turns entirely on the modifications which an uniform motion communicated to the *primum mobile* can be made to receive in passing through a train of wheels, levers, &c. and on the accuracy with which circles and straight lines can be graduated and read off) I mentioned as it occurred, and presently illustrated it by application to the well known equation between the excentric and mean anomalies in the elliptic motion of a planet.

(2.) The combination of movements which then suggested itself for this purpose, though theoretically correct, was practically liable to many objections. It happened however that I was at that time engaged in investigating the elliptic orbits of some of the most remarkable double stars; and in the course of that enquiry had continual occasion for the numerical resolution of cases of this equation in every state of the data. Finding the preliminary trials requisite for establishing a rapid conveyance of the successive approximations consume a great deal of time, even more than the approximations themselves when once effectually entered upon, I set myself to consider whether some simple contrivance free from such objections might not be found which would give me by inspection at least a first approximation to the solution, and thus prove of immediate practical utility. After one or two failures from attempting to unwind a thread from the circumference of a wheel revolving on a *fixed* center, and after rejecting as impracticable Newton's mechanical solution of the problem by the rolling of a wheel upon a plane, it occurred to me that a wheel will revolve uniformly and unwind a thread from it with a uniform motion just as well whether it revolve on its own center as an axis, or be

carried round, bodily, by the attachment of its center to the end of a revolving arm, whose length may either be permanent or adjustable by a slider. Hence arose the following construction which seems to be as simple as the nature of the problem admits.

(3.) In figure 1, *C* is a ***horizontal axis*** at right angles to the plane of the paper, on which is firmly fixed at right angles, or in the plane of rotation of the axis, a cross piece, having a bevelled groove *PQ*, in which the slider *SR* moves freely backwards and forwards until arrested and fastened by a clamping screw. Projecting forwards from the slider *SR*, and firmly rivetted into it is a short pin *B*, on which, as on an axis, the ***excentric wheel DEF*** is set, and permitted to turn freely until also arrested and clamped. These clampings are both easily performed by means of a male screw cut on the end of the pin *B* on which is fitted a female screw cut in the cross head, or ***clamping key mn.*** When this screw is loosened the slider *SR* can move in its groove and the wheel *DEF* revolve on the pin *B*, but when tightened these motions are rendered impossible, and the whole apparatus, cross-piece, slider, and excentric wheel become part of the axis *C*, and revolve with it as one mass. Thus we are enabled to adjust, first, the distance of the center *B* from the axis, and secondly, the position of a given point in the circumference of the excentric wheel with respect to a horizontal line.

(4.) The axis *C* also carries an ***index-arm CH*** furnished either with a simple ***index*** like a clock's hand, or if accuracy be required with a vernier adapted to subdivide the graduations of an ***index-circle AH*** concentric with *C*, and divided into degrees, &c. Around the excentric wheel *DEF* a thread of some fine and inextensible material (such as dentist's silk or very delicate silver wire) is wound. For this purpose the edge of the excentric wheel

must be turned truly cylindric, leaving at either side a slight elevation like a parapet to prevent the thread from slipping off. In the front edge of this elevation must be cut a small notch *F*, in which the end of the thread is to be carried off the edge of the wheel, and fastened on a pin *P* stuck (like the tuning pegs in the handle of a violin) into the face of the wheel, so as to allow of lengthening or shortening the thread a little by winding it more or less on this peg. The other end of the thread where it leaves the circumference of the wheel at *E* hangs down vertically, and has suspended to it a vertical, straight, divided scale *MN*, the divisions of which are marked and read off by the intersection of its fiducial line *MKN* with the horizontal straight edge *IKJ* at *K*, for which reading off a microscope moveable on *IJ*, and furnished with a micrometer, *might* be used should such a degree of precision be required. The straight edge *IJ* is itself also graduated for a purpose to be presently explained, and might in like manner be read off microscopically.

(5.) The axis *C* also carries on it a barrel *abc*, round which and round a fixed smooth pin *x* a string makes two or three coils (embracing both the barrel and pin in each coil), after which it is attached at both its ends to a weight *w*, which must be such as to afford a sufficient tension to the string to produce a smooth uniform friction on the barrel, and thereby to prevent the axis from turning without the continual application of a moving power, and to bring it at once to rest, without jerks, dragging, or recoil, when the power ceases to act. The power may be applied either at the circumference of the barrel by the hand, or by a handle fixed on the axis *C*, and not represented in the figure. The barrel, weight, and pin *x*, are supposed to lie *behind* the plane of the index circle *AH*, which is that of the paper—the rest of the apparatus *before* it.

(6.) Let unity (1) represent the radius of the excentric wheel, *plus* that of the thread, and e the length of the interval CB between the centers of the index circle and excentric—or the *excentricity* of the latter. Then if we draw AC parallel to the horizon, and put u for the angle ACB, the perpendicular BG will $= e \cdot \sin u$. Now, were the center of the excentric wheel preserved constantly on the level of the line AC, and that wheel itself merely made to revolve uniformly by the rotation of C about B instead of B about C, the part of the thread which would be wound up on its circumference by this rotation from the commencement of the motion would be represented by $1 \times u = u$. This then would be the quantity by which the vertical scale would, on that supposition, be raised above its original position. But in the actual case, not only is the thread so wound on the wheel, but the center B of the wheel being raised above AC by $e \sin u$, carries up with it the wheel, thread, and scale, all by the same quantity. Therefore the total elevation of the scale due to both causes acting at once will be $u + e \cdot \sin u$. If then we put A for this elevation, there will subsist between u and A the transcendental relation

$$u + e \cdot \sin u = A,$$

which is that of the problem of the excentric and mean anomalies in the elliptic motion of a planet, the arcs being reckoned from the aphelion. If we would reckon them from the perihelion, we have only to wrap the thread the other way round the excentric wheel, when the equation expressing the relation between A and u becomes

$$u - e \cdot \sin u = A.$$

(7.) It appears from what has been said, that the values of u being read off on the index circle at H, those of A will be read

off on the vertical scale at *I*. The zeros of both readings correspond to the horizontal position of the line *CB*, which position having been once well ascertained and verified, may be afterwards at any time recovered by a small level *PS* screwed on the cross-piece *PQ*, and adjusted accordingly. Both zeros are adjustable—that of the index circle by the stiff-friction motion of the index-arm *CH*, and that of the scale, first, coarsely by loosening the excentric wheel on its center, and winding or unwinding the thread on its circumference, secondly, more delicately by screwing or unscrewing the peg *p* to which its end is fastened.

(8.) The division of the vertical scale must be into 360 equal parts or degrees, the whole length from 0 to 360 being that of the circumference of a circle whose radius is 1. This length may be determined (better than by any attempt to measure the radius) by turning the axis *C* round through one complete revolution, and noting by temporary marks or dots on the scale the points intersected on the fiducial line, by the horizontal straight edge *IJ* at the two extremities of its motion. The interval between these dots is the value of 360 parts required, and must be subdivided accordingly. It is evident that whatever be the value of *e* in the equation $u + e \cdot \sin u = A$ an increase of 360° in u must correspond to 360° increase in A, so that the accuracy of this process is, theoretically speaking, independent of the position of *B* on the slider—but practically it is preferable, before executing this most essential of all the preliminary operations, to bring the center of the excentric as nearly as possible to coincidence with the axis *C*, because in that position any slight deviation from horizontality in the line *IJ* will not influence the result.

(9.) The perfect horizontality of this line is however of material import to the correct performance of the machine. It is

easily verified and secured by a spirit level and screw adjustments at one or both ends. The straight edge *IJ* itself should be graduated into equal parts, corresponding to decimals and centesimals of the radius (1), to obtain which it is only requisite to measure the length of 360° on the vertical scale as before obtained, and divide the result by $2\pi = 2 \times 3{\cdot}14159$, &c. It is likewise convenient that the straight edge should have a screw motion in a horizontal direction, by which its error of zero may be destroyed without altering its horizontal position.

(10.) The fiducial line of the vertical scale, or that whose intersection with the straight edge marks the values of A, ought to be exactly vertical when the scale hangs freely. This condition is not indeed essential to correct performance, provided it have been graduated in its inclined position, but, if satisfied, it greatly facilitates other essential adjustments, and may therefore be regarded as one of those which must be gone through. It is very easy, all that is needed being to make the lower end of the vertical scale terminate in a narrow tail-piece of lead, which being flexible, we may by bending it a little one way or other *tilt* the center of gravity of the scale, so that the fiducial line shall coincide in direction with a fine plumb line.

(11.) The values of e may be read off in two ways, either, first, by a graduation on the slider itself, (in which the fixed part may serve as a vernier to the moveable one,) or on the straight edge. In the former case the graduation must be, like that of the straight edge, into decimals and centesimals of the unit radius determined as above described, and the zero point must be ascertained as follows. Set the straight edge *IJ* and the slider both horizontal by their respective levels, adjust the index hand *H* to 0°, and bring the center *B* of the excentric wheel as nearly as may be over *C*, the center

of the horizontal axis; then make very slowly one complete revolution of that axis, noting carefully at its commencement, and end, and at every 30°, the reading off of the straight edge *IJ*, where it is intersected by the fiducial line of the vertical scale, (which should always be allowed to attain perfect rest free from lateral oscillations). If the point of intersection be found not to have varied on the straight edge, it is evident that the coincidence of *B* and *C* must have been perfect; but if otherwise, its extreme variations will mark out the diameter of the small circle which *B* continues to describe about *C*. This must be destroyed by shifting the place of *B* on the slider, and if needed, by altering the place of the slider itself on the axis (which may possibly have been originally erroneous, so as not to allow of the line described by *B* in its groove, passing through *C* at all). As soon as this is done, and the invariability of the above-mentioned intersection ascertained, a fine line must be drawn directly across the slider and its groove at each end, and thus the zero points of the divisions both of the slider and its verniers at each end are secured. The divisions should be carried along the whole length of the cross piece and along both edges of the slider, on one forwards, and on the other backwards, by which means the machine is equally adapted for positive and negative values of *e*. If the cross-piece be long enough, the machine will of course serve for values of *e* equal to or greater than unity as well as less.

(12.) If we would read off *e* on the horizontal straight edge, we must first set the index hand to 0, and then to 180°—the difference of the readings is equal to $2e$, being the diameter of the circle described by *B* transferred to the straight edge by perpendiculars to the horizon.

(13.) The only adjustment of any degree of delicacy is that of

the horizontal situation of BC, or of the cross-piece level, the line BC being imaginary and intangible. Good workmanship will of course ensure a very near approach to parallelism between this line and the two sides of the cross-piece; but if an error be still supposed to exist it may be detected by the following process: make $e = 1$, or set the slider to 1·00, and then beginning at 0° of the index read off the value of A, corresponding to equal small increments of u (for example from degree to degree) round a complete semicircle to 180°. Then, since we have

$$\frac{dA}{du} = 1 + \cos u, \text{ and } \frac{d^2A}{du^2} = -\sin u,$$

a comparison of differences and second differences will readily enable us to perceive whether any appreciable deviation from the true position exists. It is only by the differences of readings that an error in the zero of u can be separated from one in that of A, (in which is included that of the index reading at H), as the form of the equation $u + e \cdot \sin u = A$ will easily make evident.

(14.) However truly the cylindrical form of the excentric wheel is attained, if the axis of the cylinder be not parallel to that of the index circle, the thread will wind off an ellipse in place of a circular arc, of which equal portions will not correspond to equal angles of rotation. This then affords a means of detecting and rectifying such want of parallelism in the axes in question; but as its effect would be confounded with that arising from the error in the origin of u just mentioned in the last article, it will be preferable to examine through a whole revolution of the index axis (which should be perfectly horizontal) whether the thread maintains precisely the same distance from either edge of the excentric wheel as it wraps itself round it.

(15.) Supposing all these adjustments well made, the workmanship good, and the graduations such as may be executed, there seems no reason in the nature of the case why the mechanical solution afforded by the apparatus we have above described should not possess equal precision with any astronomical observation, and therefore be available in many instances where extreme nicety of computation is not required, or where several hypothetical ellipses may require to be tried in the calculation of the orbit of a comet or other celestial body. In the enquiries to which I have already alluded I found in fact a very material saving of time and trouble from the use of such an instrument, though constructed in the rudest manner from materials casually at hand.

(16.) The solution of the equation $u + e \cdot \sin u = A$ includes that of its derivative forms

$$u + a \cdot \sin (u + b) = A,$$
$$\text{and } u + a \cdot \sin u + b \cdot \cos u = A,$$

where however only sines and cosines of u are involved. If we would introduce tangents, secants, &c. we must have recourse to a modification of our mechanism. For instance, suppose the equation to be resolved were

$$u + p \cdot \tan u = A,$$

then, retaining the slider, excentric wheel, and horizontal axis C, as in the contrivance already described, let the straight edge IJ (fig. 2) instead of being permanently fixed in a horizontal position, be made to revolve on a center I vertically below C, with half the angular velocity of the index hand, which may be done either by a toothed wheel working into another of twice the diameter, or by catgut wrapping tightly round cylinders in the same proportion, and let the zero of both rotations be in the horizontal positions of CB and

IJ. Moreover, let the index circle instead of being as before concentric with the axis *C* be now described about the center *I*, and the index arm with its vernier be connected with that axis as we before supposed to be with the axis *C*.

(17.) Put u for the arc read off on this circle or the angle *XIJ*, also let $e = CB$, $BD = BE = 1$; then since the angle *GCB* is twice *XIJ*, or $= 2u$, we have

$$MJ = DE + EJ - DEM = DE + BG + KJ - DEM,$$
$$= 2u + e \,.\, \sin 2u + KL \,.\, \tan u - DEM.$$

DEM is the part of the thread included between its point of attachment to the wheel, and the zero of the scale—it is therefore an arbitrary constant. If we put c to represent it, and observe that $KL = CL - CG - GK = CI \,.\, \operatorname{cotan} u - CB \,.\, \cos 2u - 1$

$= b \,.\, \operatorname{cotan} u - e \,.\, \cos 2u - 1$, (putting b for the constant distance *CI*) we shall have for the expression of *MJ*,

$$MJ = 2u + e \,.\, \sin 2u - c + \{b \,.\, \operatorname{cotan} u - e \,.\, \cos 2u - 1\} \,.\, \tan u$$
$$= (2u + b - c) + (e - 2) \,.\, \tan u.$$

If therefore we put $MJ = 2A + b - c$; and take $e = 2p + 1$, the relation between A and u becomes

$$u + p \,.\, \tan u = A,$$

which is the equation proposed.

(18.) In order then to adopt the above construction to any given case of this equation we must set the slider so that the distance *BC* shall $= 2p + 1$. That is to say, the zero of the scale of the divisions on the slider must commence at a distance from *C* equal to the unit or radius of the excentric + that of the thread, and its graduation must be into parts *double* of the corresponding parts of p. In like manner the graduation of the vertical scale *MJ* must begin at the point where the revolving straight edge intersects

it in its horizontal position (in which also CB must be adjusted to be accurately horizontal,) and the parts of this scale must be double of the corresponding parts of A, that is, ***double*** degrees of u. Each part therefore must be one 180th part of the circumference of the circle DEF, as measured by the unwinding of the thread in the mode explained in (Art. 8.)

(19.) If the equation proposed were the rather more general one

$$u + p \,.\, \tan u + q \,.\, \sec u = A,$$

we might employ the same construction slightly modified by making the straight edge IJ revolve attached to an arm at right angles to the axis of the lower wheel, so that the motion of rotation of IJ shall as before be uniform, and half as swift as that of CB, but its direction not passing through the center Q of its revolutions. All other things remaining as in (Art. 16.), let the perpendicular $QI = f$ then will the value of MJ be as before,

$$MJ = 2u + e \sin 2u + (CL - CG - GK.) \,.\, \tan u,$$

but in this case we have

$$CL = f \,.\, \operatorname{cosec} u + b \,.\, \operatorname{cotan} u,$$

so that the value of MJ becomes

$$\begin{aligned} MJ &= 2u + e \,.\, \sin 2u + \{f \,.\, \operatorname{cosec} u + b \,.\, \operatorname{cotan} u + e \,.\, \cos 2u - 1\} \,.\, \tan u, \\ &= (2u + b) + (e-1) \,.\, \tan u + f \,.\, \sec u, \end{aligned}$$

that is, putting $MJ = 2A + b$; $e = 2p + 1$; $f = 2q$,

$$A = u + p \,.\, \tan u + q \,.\, \sec u.$$

(20.) It is needless to enlarge on the adjustment of these contrivances, which are merely introduced as specimens of the variations which a trifling change in the construction of our mechanism is capable of making in the form of the equations resolved. I will only observe here, that as ovals can now be turned of almost

any excentricity, which shall scarcely deviate perceptibly from the true elliptic figure, so, in all these and similar cases, by substituting elliptic for circular *excentric wheels*, the arc u instead of being directly related to the sines, cosines, tangents, &c., involved in the equations may, without the slightest increase of mechanism, or any additional difficulty in the process of solution, be replaced by a transcendent of that form which depends on the rectification of the ellipse.

(21.) It is almost needless to mention that any mechanical contrivance which converts a uniform motion u into another not uniform, but varying according to any function $\phi(u)$ of the former, affords either a solution of the equation $\phi(u) = A$, or a tabulation of the values of $\phi(u)$, just as we please. In the one case we have only to arrest the motion at equal intervals of the scale on which the graduation of A is engraved, and read off the graduation of that on which u is represented. In the latter the movements must be arrested at equal intervals of u, and the values of A read off. Thus the tabulations of the direct and inverse functions proceed, *pari passu*.

(22.) It is not my intention in this paper to enter at large into the general question of the representation of analytical functions by continued motion, though perhaps I may take a future opportunity of so doing. I will only here consider one other case, by which, without any great complication of machinery, the principle I have above adopted may be extended to equations containing several transcendental relations, such as

$$u + p \,.\, \sin mu + q \,.\, \sin nu = A,$$

and others of the same nature. Suppose, instead of attaching our excentric wheel to a point in the revolving arm BC we attach it to a second revolving arm, whose center of rotation occupies the

point which that of the excentric itself occupied in the original construction, and let this second arm have a velocity of rotation in a constant ratio to that of the first, a condition attainable by contrivances to be presently considered. In that case our construction will be as in fig. (4), respecting which figure we will establish the following notation.

$CB = c$; $BB' = e$; $B'E' = 1$; angle $ACB = \alpha u$; $DBB' = \beta u$; $MK = x$, and $D'E'M = b$;

when we have

$$MK = x = D'E'K - D'E'M = D'E' + B'G - D'E'M,$$
$$= (\alpha + \beta) u + B'T + BV - c,$$
$$= (\alpha + \beta) u - c + e \cdot \sin \alpha u + e' \cdot \sin (\alpha + \beta) u,$$

assume therefore $\alpha = m$; $\alpha + \beta = n$; $e = np$; $e' = nq$; $x = nA - c$, and the relation between u and A will be that proposed, viz

$$u + p \cdot \sin mu + q \cdot \sin nu = A.$$

This equation, it will be observed, can always be so prepared as to make m and n integers, or, if we prefer it, fractions, whose denominators are integers. The form however which will require the least apparatus of wheels, and into which it is easily transformed, is the following:

$$A = u + p \cdot \sin u + q \cdot \sin nu,$$

in which n is less than unity; for in this state of the equation the first mover may be applied at once to the axis C, which, as in the former construction, may carry an index arm reading off u in degrees on a circle concentric with it. The whole difficulty then is reduced to the solution of a mechanical problem. To communicate to the arm BB' revolving on a center B, attached to another revolving arm CB, a rotation having a given ratio of velocity

to that with which the latter arm itself revolves, it being understood that the point of attachment of the center B is to be capable of adjustment to a greater or less distance from C; a condition which excludes the use of toothed wheel work. The following is the simplest construction which has occurred to me for accomplishing this purpose.

(23.) The horizontal index axis C is surrounded by a grooved wheel $g\,h\,i$, (figs. 4, 5,) which lies behind the index plate (not represented in those figures) and the cross-piece and slider, and is not attached to the axis so as to turn with it, but on the contrary is fixed to the index plate by screws so as to prevent all rotation. The end f of the cross-piece is penetrated by an axis f seen in projection in fig. 5, but lengthwise in fig. 6, whose extremities (to avoid shake and loosening) are pivoted in a bifurcated and recurved prolongation of the cross-piece, which is seen in fig. 6 at $e\,f$. This axis carries on it two wheels also grooved, $c\,d$, $c'\,d'$, both firmly united to the axis, and therefore incapable of moving unless together as one wheel. A string or band passes round the groove in $g\,h$ and $c\,d$, so that when the axis C is made to revolve, and therefore the wheel $c\,d$ is carried round $g\,h$, the latter remaining immoveable, the *relative* rotation of the one wheel about the other will wrap and unwrap the string round the groove $g\,h$, and thus produce a rotation of the wheel $c\,d$ on its axis f, just as it would do if all the rest of the apparatus stood still, and the wheel $g\,h$ alone was turned the other way round the axis C. Thus a rotation equal and contrary to that of C, or, if the wheels $g\,h$, $c\,d$, be of unequal size, in any constant ratio to the latter rotation, is communicated to the wheels $c\,d$, $c'\,d'$. From the latter of these let a string be led round a groove in the wheel $a\,b$, whose axis is the pin B on which the second slider revolves, and which

slider is firmly screwed on the face of the wheel, so as to revolve with it. The distance between the axes f and B of these wheels being changeable a re-entering string cannot be used, but it must, after making a complete circumvolution of both, be fastened off at each of its extremities to pegs, and brought by them into the requisite state of tension. The rotation of cd will thus be transferred to ab, increased or diminished in any ratio according to the ratio of the diameters of the wheels $c'd'$ and ab, which must be adjusted accordingly. It is evident that this construction accomplishes the end in question, which might indeed be accomplished without the use of the wheels cd, $c'd'$, were it not necessary to allow the center B completely to attain and even pass across C. It is very likely that other and still simpler modes of accomplishing the same object will suggest themselves to others.

To accompany Sir J. Herschels paper

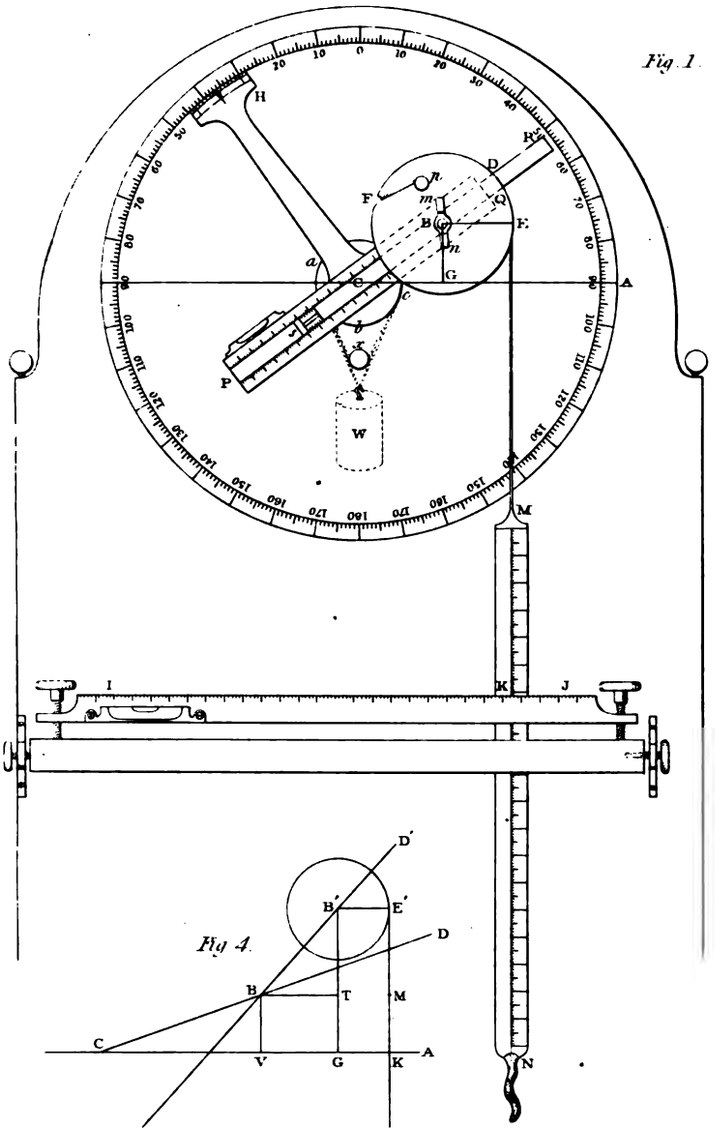


Fig. 1.

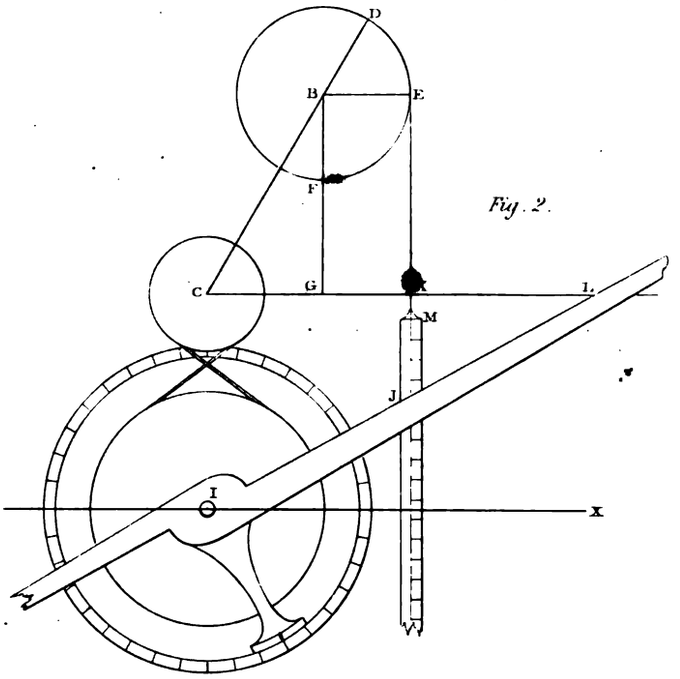


Fig. 2.

Fig 4.

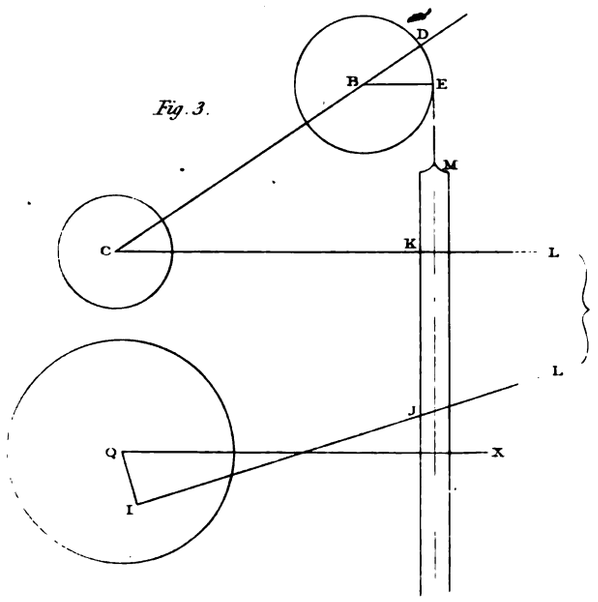


Fig. 3.

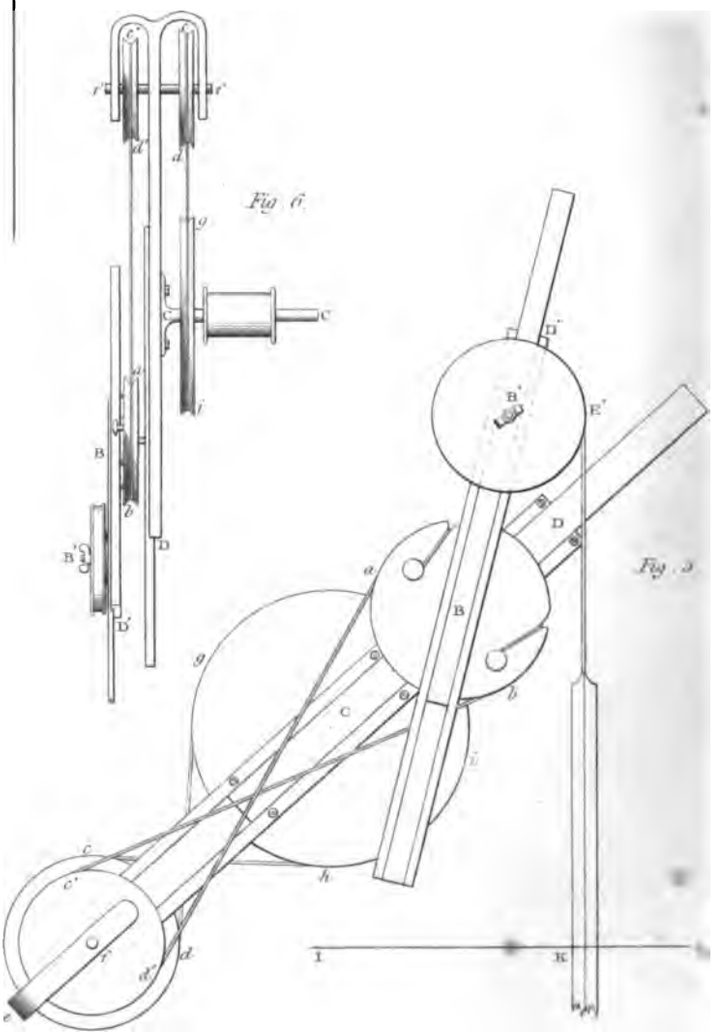


Fig. 5

Fig. 6.

J.W. Lowry Sculp

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